MSW MANAGEMENT -**Multidisciplinary, Scientific Work and Management Journal**
ISSN: 1053-7899
Vol. 35 Issue 1, Jan-June 2025, Pages: 157-168

A RESULT RELATED TO BRÜCK CONJECTURE AND LINEAR DIFFERENTIAL POLYNOMIAL

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Abstract: In connection to Brück conjecture we improve a uniqueness problem for entire functions that share a polynomial with linear differential polynomials.

Keywords and phrases: Brück conjecture, entire function, polynomial sharing, linear differential polynomial

2020 Mathematics Subject Classification : 30D35

1. Introduction, Definitions and Results

Let f, g and a be entire functions in the open complex plane $\mathbb{C}$. If $f-a$ and $g-a$ have the same set of zeros with the same multiplicities, then we say that f and g share the function a CM (counting multiplicities). If, in particular, a is a constant, then we say that f and g share the value a CM.

For an entire function f, $M(r,f) = \max_{|z|=r} |f(z)|$ denotes the *maximum modulus function* of f. If the Taylor expansion of f is $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then the power series $\sum_{n=0}^{\infty} |a_n| r^n$ converges for every $r > 0$ and so for any given $r > 0$, we have $\lim_{r\to\infty} |a_n| r^n = 0$. Hence the maximum term $\mu(r,f) = \max_{n\geq 0} |a_n| r^n$ is well defined.

Also we define $\nu(r,f)$, the *central index* of f, as the greatest exponent m such that $\mu(r,f) = |a_m| r^m$ {see p.50 [7]}.

Then

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r}$$

and

$$\lambda(f) = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r}$$

are respectively called the *order* and *lower order* of f {see p.51 [7]}.

Also

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r}$$

and

$$\lambda_2(f) = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log \nu(r,f)}{\log r}$$

are respectively called the *first iterated order* or *hyper-order* and *first iterated lower order* or *lower hyper-order* of f {see Lemma 2 in [3]}.

In 1977 L. A Rubel and C. C. Yang [8] first considered the uniqueness problem of values sharing by a nonconstant entire function with its first derivative. This work of Rubel and Yang inspired a lot of researchers to explore such type of problems and extend it to different directions. In this direction, in 1996 R. Brück [2] proposed the following conjecture.

Brück's Conjecture: Let f be a nonconstant entire function such that $\sigma_2(f) < \infty$ and $\sigma_2(f) \notin \mathbb{N}$. If f and $f^{(1)}$ share a finite value a CM, then $f^{(1)} - a = c(f - a)$, where c is a nonzero constant.

Though Brück himself resolved the conjecture for $a = 0$, the case $a \neq 0$ is not yet fully resolved.

For an entire function of finite order, G. G. Gundersen and L. Z. Yang [5] and L. Z. Yang [10] resolved and generalised Brück conjecture and proved the following results.

Theorem A. [5] *Let f be a nonconstant entire function of finite order. If f and $f^{(1)}$ share one finite value a CM, then $f^{(1)} - a = c(f - a)$ for some nonzero constant c.*

Theorem B. [10] *Let f be a nonconstant entire function of finite order. If f and $f^{(k)}$ share one finite value a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c.*

In 2004 J. P. Wang [9] extended Theorem B by considering polynomial sharing with its higher order derivatives and improved in the following manner.

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Theorem C. [9] *Let f be a nonconstant entire function of finite order and a be a nonconstant polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f - a)$ for some nonzero constant c.*

Afterwards Z. X. Chen and K. H. Shon [4] and I. Lahiri and S. Das [6] extended Theorem A to a class of entire functions of unrestricted order and proved the following theorems.

Theorem D. [4] *Let f be a nonconstant entire function with $\sigma_2(f) < \frac{1}{2}$. If f and $f^{(1)}$ share a finite value a CM, then $f^{(1)} - a = c(f - a)$, where c is a nonzero constant.*

Theorem E. [6] *Let f be a nonconstant entire function with $\lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $a = a(z)$ is a polynomial. If f and $f^{(k)}$ share a CM, then $f^{(k)} - a = c(f - a)$, where c is a nonzero constant.*

In the paper, the aim is to improve Theorem C, Theorem D and Theorem E by considering the following problems:

(i) Replacement of shared value by shared polynomial;
(ii) Replacement of higher derivatives by linear differential polynomial.

We now state the main result of the paper.

Theorem 1.1. *Let f be a nonconstant entire function such that $\sigma(f) \neq 1, \lambda_2(f) < \frac{1}{2}$ and $\sigma_2(f) < \infty$. Suppose that $a = a(z)$ is a polynomial.*
Let $L(f) = a_0 f + a_1 f^{(1)} + \cdots + a_k f^{(k)}$, where $k(\geq 1)$ is an integer and $a_0, a_1, \ldots, a_k(\neq 0)$ are constants.
If f and $L(f)$ share a CM, then $L(f) - a = c(f - a)$, where c is a nonzero constant.

Following example shows that the condition $\sigma(f) \neq 1$ is essential.

Example 1. *Let $f(z) = e^z + z$ and $L(f) = f^{(2)} - 2f^{(1)} + f$. Then f and $L(f)$ share z CM but $L(f) - z = -2e^{-z}(f - z)$, where f satisfies $\sigma(f) = 1$.*

2. Lemmas

In this section we present some necessary lemmas.

Lemma 2.1. {p.5 [7]} *Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then for any $\delta > 1$, there exists $R > 0$ such that $g(r) \leq h(r^\delta)$ holds for $r > R$.*

Lemma 2.2. {p.9 [7]} *Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_0 (b_n \neq 0)$ be a polynomial of degree n. Then for every $\varepsilon(> 0)$ there exists $R(> 0)$ such that for all $|z| = r > R$ we get*

$$(1 - \varepsilon)|b_n| r^n \leq |P(z)| \leq (1 + \varepsilon)|b_n| r^n.$$

Lemma 2.3. {p.51 [7]} *Let f be a transcendental entire function. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure such that for $|z| = r \notin [0, 1] \cup E$ and $|f(z)| = M(r, f)$ we get*

$$\frac{f^{(k)}(z)}{f(z)} = (1 + o(1)) \left\{ \frac{\nu(r, f)}{z} \right\}^k$$

for $k = 1, 2, 3, \ldots, n$, where n is a positive integer.

Let $h(z)$ be a nonconstant function subharmonic in the open complex plane $\mathbb{C}$ and let

$$A(r) = A(r,h) = \inf_{|z|=r} h(z) \text{ and } B(r) = B(r,h) = \sup_{|z|=r} h(z).$$

Then the *order* $\sigma(h)$ and the *lower order* $\lambda(h)$ of h are defined respectively by

$$\sigma(h) = \limsup_{r\to\infty} \frac{\log B(r,h)}{\log r}$$

and

$$\lambda(h) = \liminf_{r\to\infty} \frac{\log B(r,h)}{\log r}.$$

The *upper logarithmic density* and the *lower logarithmic density* of $E \subset [1,\infty)$ are respectively defined by

$$\overline{\text{logdens}}(E) = \limsup_{r\to\infty} \frac{\int_1^r \frac{\chi_E(t)}{t}dt}{\log r}$$

and

$$\underline{\text{logdens}}(E) = \liminf_{r\to\infty} \frac{\int_1^r \frac{\chi_E(t)}{t}dt}{\log r},$$

where χ_E be the *characteristic function* of E.

The quantity $\lim_{r\to\infty} \int_1^r \frac{\chi_E(t)}{t}dt$ defines the *logarithmic measure* of E. It is easy to note that if $\overline{\text{logdens}}(E) > 0$, then E has infinite logarithmic measure.

Lemma 2.4. [1] *Let $h(z)$ be a nonconstant subharmonic function in the open complex plane $\mathbb{C}$ of lower order $\lambda, 0 \le \lambda < 1$. If $\lambda < \beta < 1$, then*

$$\overline{\text{logdens}}\{r : A(r) > (\cos\beta\pi)B(r)\} \ge 1 - \frac{\lambda}{\beta},$$

where $A(r) = \inf_{|z|=r} h(z)$ and $B(r) = \sup_{|z|=r} h(z)$.

3. Proof of Theorem 1.1

Proof. By the hypothesis we have

$$\frac{L(f) - a}{f - a} = e^A, \tag{3.1}$$

where A is an entire function.

If A is a constant, then the result holds clearly. So we suppose that A is a nonconstant entire function and consider the following two cases.

Case 1. Let $\sigma(f) < \infty$. Then from (3.1) we get that A is a polynomial. If $\sigma(f) < 1$, then (3.1) implies that A is a constant. So $\sigma(f) > 1$ and therefore f is a transcendental entire function.

Now we suppose that A is a nonconstant polynomial.

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Now for any z with $|f(z)| = M(r,f)$ we get by Lemma 2.2 (choosing $\varepsilon = \frac{1}{2}$)

$$\left|\frac{a(z)}{f(z)}\right| \leq \frac{M(r,a)}{M(r,f)} \leq \frac{\frac{3}{2}|\alpha| r^{\deg a}}{M(r,f)} \to 0 \tag{3.2}$$

as $r \to \infty$, where α is the leading coefficient of the polynomial $a(z)$.

Now by Lemma 2.3 there exists $E \subset [1,\infty)$ with finite logarithmic measure such that for $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$ we get

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu(r,f)}{z}\right)^j (1+o(1)), \tag{3.3}$$

for $j = 1, 2, \ldots n$, where n is a positive integer.

Now for all z with $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$ we get by (3.3)

$$\frac{L(f)}{f} = a_0 + \sum_{j=1}^{k} a_j \left(\frac{\nu(r,f)}{z}\right)^j (1+o(1)). \tag{3.4}$$

From (3.1) we get

$$e^A = \frac{\frac{L(f)}{f} - \frac{a}{f}}{1 - \frac{a}{f}}. \tag{3.5}$$

Now for all z with $|z| = r \notin E \cup [0,1]$ and $|f(z)| = M(r,f)$, noting that $\sigma(f) > 1$, we get by (3.3), (3.4) and (3.5)

$$e^A = a_0 + a_k \left(\frac{\nu(r,f)}{z}\right)^k (1+o(1)). \tag{3.6}$$

Now from (3.6) we get for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$\begin{aligned} |A(z)| &= |\log e^{A(z)}| \\ &= \left|\log \left(\frac{\nu(r,f)}{z}\right)^k\right| + o(1) \\ &= |k \log \nu(r,f) - k \log z| + o(1) \\ &\leq k \log \nu(r,f) + k \log r + 6k\pi \\ &< 2k(\sigma(f)+1) \log r + 6k\pi. \end{aligned} \tag{3.7}$$

Also by Lemma 2.2 (choosing $\varepsilon = \frac{1}{2}$) we obtain for all large $|z| = r$

$$\frac{1}{2}|\alpha| r^{\deg A} \leq |A(z)|, \tag{3.8}$$

where α is the leading coefficient of A.

Now the equations (3.6) and (3.7) together imply $\deg A = 0$ and so A is a constant, which is a contradiction.

Case 2. Let $\sigma(f) = \infty$. We now consider the following two subcases.

Subcase 2.1. Let A be a nonconstant polynomial. Then from (3.6) we get for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$|A(z)| \leq k \log \nu(r,f) + k \log r + 6k\pi. \tag{3.9}$$

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Then from (3.8) and (3.9) we obtain for all large $|z| = r \notin [0,1] \cup E$ with $|f(z)| = M(r,f)$

$$\frac{1}{2}|\alpha| r^{\deg A} \leq k \log \nu(r,f) + k \log r + 6k\pi. \tag{3.10}$$

Hence by Lemma 2.1 for given δ, $1 < \delta < \frac{3}{2}$ and (3.10), we get for all large values of r

$$\frac{1}{2}|\alpha| r^{\deg A} \leq k \log \nu(r^{\delta},f) + k\delta \log r + 6k\pi$$

and so

$$r^{\deg A}\left(\frac{1}{2}|\alpha| - \frac{k\delta \log r}{r^{\deg A}}\right) \leq k \log \nu(r^{\delta},f) + 6k\pi.$$

This implies $\deg A \leq \delta\lambda_2(f) < \frac{\delta}{2} < \frac{3}{4} < 1$, a contradiction. Therefore A is a constant.

Subcase 2.2. Let A be a transcendental entire function. Since for an entire function $A(z)$, $h(z) = \log|A(z)|$ is a subharmonic function in $\mathbb{C}$, and also from (3.1) we get $\lambda(h) = \lambda_2(A) \leq \lambda_2(f) < \frac{1}{2}$.

Suppose that $H = \{r : A(r) > (\cos\beta\pi)B(r)\}$, where $A(r) = \inf_{|z|=r} \log|f(z)|$, $B(r) = \sup_{|z|=r} \log|f(z)|$ and $\beta \in \left(\lambda_2(A), \frac{1}{2}\right)$.

Then by Lemma 2.4 we see that $\overline{\text{logdens}} H > 0$, i.e., H has infinite logarithmic measure. Also by Lemma 2.3 for $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r,f)$ we get

$$\frac{f^{(k)}(z)}{f(z)} = (1 + o(1))\left(\frac{\nu(r,f)}{z}\right)^k. \tag{3.11}$$

Now by (3.2), (3.5) and (3.11) we get for all large $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r,f)$

$$e^{A(z)} = a_0 + a_k \left(\frac{\nu(r,f)}{z}\right)^k (1 + o(1))$$

and so

$$\begin{aligned} |A(z)| &= \left|\log e^{A(z)}\right| \\ &= \left|\log\left(\frac{\nu(r,f)}{z}\right)^k\right| + o(1) \\ &= |k \log \nu(r,f) - k \log z| + o(1) \\ &\leq k \log \nu(r,f) + k \log r + 6k\pi \\ &< 2kr^{\sigma_2(f)+1}. \end{aligned} \tag{3.12}$$

Now by Lemma 2.1, there exists a constant c, $0 < c < 1$ such that for all z satisfying $|z| = r \in H \backslash \{[0,1] \cup E\}$ with $|f(z)| = M(r,f)$, we have

$$(M(r,A))^c < |A(z)|. \tag{3.13}$$

Now by (3.12) and (3.13), we get

$$\frac{(M(r,A))^c}{r^{\sigma_2(f)+1}} < 2k. \tag{3.14}$$

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This is impossible because A is transcendental and so $\frac{(M(r,A))^c}{r^{\sigma_2(f)+1}} \to \infty$ as $r \to \infty$. This proves the theorem. □

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