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Bicomplex Matrix Analysis Via Idempotent Decomposition

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Abstract: This paper investigates algebraic properties of bicomplex matrices via their idempotent decomposition. After presenting the necessary preliminaries, we establish several results concerning fundamental matrix operations. In particular, it is shown that the transpose, cofactor, determinant, and adjoint of the idempotent components coincide with the corresponding operations applied component-wise. Key structural properties of adjoint matrices are derived, including the relation $\text{adj.}(AB) = \text{adj.}(B)\ \text{adj.}(A)$, along with the preservation of Hermitian structure in each idempotent component. The concepts of singular and strictly singular bicomplex matrices are introduced and characterized with illustrative examples. Notably, a bicomplex matrix is singular if and only if its adjoint is singular. Further results on products of bicomplex matrices are obtained, providing conditions under which zero products imply singularity. The notion of orthogonal bicomplex matrices is also introduced and examined. It is shown that their determinants belong to the set $\{1, -1, i, -i\}$, and that this class is closed under multiplication. Moreover, idempotent component matrices are shown to be orthogonal.

These findings enhance the structural understanding of bicomplex matrices and provide a foundation for further developments in bicomplex linear algebra.

1. Introduction

The theory of bicomplex numbers has attracted considerable attention due to its rich algebraic structure and its connections with complex and hypercomplex analysis. As an extension of complex numbers, bicomplex numbers provide a natural framework for studying higher-dimensional algebraic systems. In recent years, this framework has been increasingly explored in various areas of mathematics, particularly in linear algebra and operator theory.

Matrices over bicomplex numbers, known as bicomplex matrices [2,3,8], form an important class of objects that extend many classical concepts from real and complex matrix theory. However, due to the presence of zero divisors and the non-trivial algebraic structure of bicomplex numbers, several properties of matrices require careful reinterpretation. One of the most effective tools in this context is the idempotent decomposition, which allows bicomplex matrices to be expressed in terms of simpler components and facilitates the study of their structural properties.Motivated by these considerations, the present work focuses on the investigation of algebraic properties of bicomplex matrices through their idempotent components. In particular, we analyze fundamental matrix operations such as transpose, cofactor, determinant, and adjoint, and examine how these operations behave under idempotent decomposition. Furthermore, we introduce the notions of singular and strictly singular bicomplex matrices and provide their characterizations with suitable examples. Several results concerning adjoint matrices and products of bicomplex matrices are established, highlighting conditions under which singularity arises. In addition, the concept of orthogonal bicomplex matrices is developed and studied, along with some of their fundamental properties.This study is carried out in a systematic manner and is expected to provide a useful foundation for future research in bicomplex linear algebra and related areas.The structure of this paper is as follows. In 1, we introduce the essential background and preliminary concepts. 2 is devoted to the main results of the study, where new findings concerning bicomplex matrices and their structural characteristics are developed. 3 contains conclusion part.

Background and Essentials

This section presents the essential preliminary material in an organized way. Important symbols, properties, and results related to bicomplex numbers and matrices are collected here, along with key distinctions between the structure of bicomplex space and that of the usual complex space.

The system of bicomplex numbers was originally introduced by Corrado Segre in 1892 [12], where it was shown to form a commutative algebra over the field of real numbers. Bicomplex numbers can be regarded as a natural extension of complex numbers, while hyperbolic numbers arise as a special case when $x_1 = 0 = x_2$ in their real representation. Their algebraic features are both fundamental and useful [10].

Although the bicomplex algebra is commutative, it contains zero divisors, implying that multiplicative inverses do not exist for every nonzero element. In contrast, quaternion algebra is a division algebra, where every nonzero element is invertible, but multiplication is not commutative. This sharp algebraic distinction plays an important role in the development of analytical frameworks for these structures [5,6,7].

The commutative nature of bicomplex numbers therefore distinguishes their algebra from that of quaternions in a fundamental way [11].

Definition 1.1 (**Bicomplex numbers:**). A bicomplex number may be expressed as

$$\phi = x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4,$$

where all x_i are real, $1 \le i \le 4$, and the imaginary units satisfy $i_1^2 = i_2^2 = -1$ together with $i_1 i_2 = i_2 i_1$.

The notation $\mathbb{C}_2$ denotes the set of all bicomplex numbers, while $\mathbb{C}_1$ and $\mathbb{C}_0$ represent the sets of complex and real numbers, respectively. The bicomplex space $\mathbb{C}_2$ admits three equivalent and widely used representations:

$$\begin{aligned}
\mathbb{C}_2 &= \left\{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 \ :\ x_1, x_2, x_3, x_4 \in \mathbb{C}_0\right\},\\
\mathbb{C}_2 &= \{r_1 + i_2 r_2 \ :\ r_1, r_2 \in \mathbb{C}_1\},\\
\mathbb{C}_2 &= \left\{\phi^- e_1 + \phi^+ e_2 \ :\ \phi^-, \phi^+ \in \mathbb{C}_1\right\}, \text{ see [4]}.
\end{aligned}$$

These are referred to as the real form, complex form, and idempotent form of $\mathbb{C}_2$, respectively. Since $\mathbb{C}_2$ contains zero divisors, it does not constitute a field; however, it forms a commutative modified algebra over $\mathbb{C}_1$ with a suitably norm [9]. The space $\mathbb{C}_2$ has exactly four idempotent elements: $0, 1, e_1, e_2$. The two nontrivial idempotent elements are given by

$$e_1 = \frac{1 + i_1 i_2}{2}, \quad e_2 = \frac{1 - i_1 i_2}{2}.$$

They satisfy $e_1 e_2 = 0 = e_2 e_1$, $(e_i)^n = e_i$ for $i = 1,2$, and $e_1 + e_2 = 1$, $e_1 - e_2 = i_1 i_2$. Moreover, e_1 and e_2 are linearly independent elements of $\mathbb{C}_2$ over $\mathbb{C}_1$.

Any bicomplex number $\phi = r_1 + i_2 r_2$ can be uniquely written using the idempotent basis $\{e_1, e_2\}$ as

$$\phi = (r_1 - i_1 r_2) e_1 + (r_1 + i_1 r_2) e_2.$$

The quantities $(r_1 - i_1 r_2)$ and $(r_1 + i_1 r_2)$ are referred to as the idempotent components of ϕ, and are denoted by ϕ^- and ϕ^+, respectively. Hence, ϕ can be expressed in the form $\phi = \phi^- e_1 + \phi^+ e_2$, where $\phi^- = r_1 - i_1 r_2$ and $\phi^+ = r_1 + i_1 r_2$. This convention follows the notation of Prof. R.K. Srivastava [13].

Remark 1.2. The algebraic operations of addition and multiplication in $\mathbb{C}_2$ are most effectively expressed through the idempotent decomposition. In particular, for $\phi, \psi \in \mathbb{C}_2$, we obtain

$$\begin{aligned}
\phi\psi &= (\phi^- \psi^-) e_1 + (\phi^+ \psi^+) e_2,\\
\phi + \psi &= (\phi^- + \psi^-) e_1 + (\phi^+ + \psi^+) e_2.
\end{aligned}$$

It is worth noting that the product of two bicomplex numbers may also be written in terms of their complex components as

$$\begin{aligned}
\phi\psi &= (r_1 + i_2 r_2)(w_1 + i_2 w_2)\\
&= (r_1 w_1 - r_2 w_2) + i_2 (r_1 w_2 + r_2 w_1)
\end{aligned}$$

Furthermore, Scalar multiplication is defined component wise in the idempotent basis:

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Moreover, scalar multiplication acts component wise on the idempotent expansion:

$$\begin{aligned}\alpha\cdot\phi &= \alpha\cdot(\phi^- e_1+\phi^+ e_2)\\ &= \alpha\phi^- e_1+\alpha\phi^+ e_2.\end{aligned}$$

Definition 1.3 (**Norm of a bicomplex number:**). For an element $\phi\in\mathbb{C}_2$, the associated norm can be equivalently expressed in the following three forms:

$$\begin{aligned}\|\phi\| &= (|x_1|^2+|x_2|^2+|x_3|^2+|x_4|^2)^{\frac{1}{2}}, \quad \text{(real form)}\\ &= (|r_1|^2+|r_2|^2)^{\frac{1}{2}}, \quad \text{(complex form)}\\ &= \left(\frac{|\phi^-|^2+|\phi^+|^2}{2}\right)^{\frac{1}{2}}, \quad \text{(idempotent form)}\end{aligned}$$

Thus, these three equivalent representations of the norm on $\mathbb{C}_2$ are commonly referred to as the real form, complex form, and idempotent form, respectively. Furthermore, the norm satisfies a multiplicative property for all $\phi,\psi\in\mathbb{C}_2$:

$$\|\phi\psi\|\le\sqrt{2}\,\|\phi\|\,\|\psi\|.$$

Because of the above inequality, the bicomplex algebra $\mathbb{C}_2$, when equipped with the ordinary norm, becomes a modified Banach algebra.This relation is often referred to as the modified multiplicative inequality for the standard norm [9].

Definition 1.4 (**Characterization of Singular Elements:**). Let $\phi=r_1+i_2r_2$ be an aelement in $\mathbb{C}_2$. Then ϕ is called non-singular if there exists some $\psi=w_1+i_2w_2\in\mathbb{C}_2$ such that $\phi\psi=1$. If no such ψ exists, then ϕ is referred to as singular. It is well known that $\mathbb{C}_2$ contains infinitely many elements that lack multiplicative inverses. The entire set of such singular elements in $\mathbb{C}_2$ will be denoted by $\mathbb{O}_2$.

A bicomplex number $\phi=r_1+i_2r_2$ is singular precisely if and only if $|r_1^2+r_2^2|=0$. For $\phi=r_1+i_2r_2$ and $\psi=w_1+i_2w_2$ in $\mathbb{C}_2$, the product $\phi\psi$ is singular if and only if at least one of ϕ and ψ is singular.

Definition 1.5 (**Conjugate of a bicomplex number:**). In analogy with the conjugate operation in $\mathbb{C}_1$, a number in $\mathbb{C}_2$ admits several natural notions of conjugation. Among the most commonly used are three conjugations, associated with the units i_1, i_2, and i_1i_2, denoted by $\overline{\phi}$, $\tilde{\phi}$, and $\phi^\sharp$, respectively. For $\phi=x_1+i_1x_2+i_2x_3+i_1i_2x_4=z_1+i_2z_2$, one obtains:

$$\begin{aligned}\overline{\phi} &= (x_1-i_1x_2)+i_2(x_3-i_1x_4),\\ \tilde{\phi} &= (x_1+i_1x_2)-i_2(x_3+i_1x_4),\\ \phi^\sharp &= (x_1-i_1x_2)-i_2(x_3-i_1x_4).\end{aligned}$$

It is very easy to see that $\overline{\phi}=(\overline{z_1}+i_2\overline{r_2})=\overline{(\phi^+)}e_1+\overline{(\phi^-)}e_2$, $\tilde{\phi}=(r_1-i_2r_2)=(\phi^+)e_1+(\phi^-)e_2$, and $\phi^\sharp=(\overline{r_1}-i_2\overline{r_2})=\overline{(\phi^-)}e_1+\overline{(\phi^+)}e_2$. Moreover, if $\psi=y_1+i_1y_2+i_2y_3+i_1i_2y_4=w_1+i_2w_2\in\mathbb{C}_2$ then

$$\begin{aligned}\overline{(\phi\psi)} &= \overline{\phi}\times\overline{\psi},\\ \widetilde{(\phi\psi)} &= \tilde{\phi}\times\tilde{\psi},\\ (\phi\psi)^\sharp &= \phi^\sharp\times\psi^\sharp \text{ (see [1]).}\end{aligned}$$

These identities demonstrate that multiplication in $\mathbb{C}_2$ remains consistent with the three principal conjugations of bicomplex numbers.

Definition 1.6 (**Bicomplex Matrix:**). A matrix A whose entries are bicomplex numbers, is called a bicomplex matrix [4]. Explicitly,

$$A=\begin{bmatrix}\phi_{11}&\phi_{12}&\cdots&\phi_{1n}\\ \phi_{21}&\phi_{22}&\cdots&\phi_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \phi_{m1}&\phi_{m2}&\cdots&\phi_{mn}\end{bmatrix}, \phi_{pq}\in\mathbb{C}_2; m,n\in\mathbb{N}, 1\le p\le m,\ 1\le q\le n.$$

Since bicomplex numbers $\phi_{pq}=\alpha_{pq}+i_1\beta_{pq}+i_2\gamma_{pq}+i_1i_2\delta_{pq}=z_{pq}+i_2w_{pq}=\phi_{pq}{}^-e_1+\phi_{pq}{}^+e_2$, $1\le p\le m$, $1\le q\le n$, admit three canonical forms of representation, a bicomplex matrix can be expressed in the following ways:

1. Real Representation of A:

$$\begin{aligned}A =& \begin{bmatrix}\alpha_{11}&\alpha_{12}&\cdots&\alpha_{1n}\\ \alpha_{21}&\alpha_{22}&\cdots&\alpha_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \alpha_{m1}&\alpha_{m2}&\cdots&\alpha_{mn}\end{bmatrix}+i_1\begin{bmatrix}\beta_{11}&\beta_{12}&\cdots&\beta_{1n}\\ \beta_{21}&\beta_{22}&\cdots&\beta_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \beta_{m1}&\beta_{m2}&\cdots&\beta_{mn}\end{bmatrix}\\ &+i_2\begin{bmatrix}\gamma_{11}&\gamma_{12}&\cdots&\gamma_{1n}\\ \gamma_{21}&\gamma_{22}&\cdots&c_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \gamma_{m1}&\gamma_{m2}&\cdots&\gamma_{mn}\end{bmatrix}+i_1i_2\begin{bmatrix}\delta_{11}&\delta_{12}&\cdots&\delta_{1n}\\ \delta_{21}&\delta_{22}&\cdots&\delta_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \delta_{m1}&\delta_{m2}&\cdots&\delta_{mn}\end{bmatrix}.\end{aligned}$$

2. Complex Representation of A:

$$A=\begin{bmatrix}z_{11}&z_{12}&\cdots&z_{1n}\\ z_{21}&z_{22}&\cdots&z_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ z_{m1}&z_{m2}&\cdots&z_{mn}\end{bmatrix}+i_2\begin{bmatrix}w_{11}&w_{12}&\cdots&w_{1n}\\ w_{21}&w_{22}&\cdots&w_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ w_{m1}&w_{m2}&\cdots&w_{mn}\end{bmatrix}.$$

3. Idempotent Representation of A:

$$A = \begin{bmatrix}\phi^-{}_{11}&\phi^-{}_{12}&\cdots&\phi^-{}_{1n}\\ \phi^-{}_{21}&\phi^-{}_{22}&\cdots&\phi^-{}_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \phi^-{}_{m1}&\phi^-{}_{m2}&\cdots&\phi^-{}_{mn}\end{bmatrix}e_1+\begin{bmatrix}\phi^+{}_{11}&\phi^+{}_{12}&\cdots&\phi^+{}_{1n}\\ \phi^+{}_{21}&\phi^+{}_{22}&\cdots&\phi^+{}_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ \phi^+{}_{m1}&\phi^+{}_{m2}&\cdots&\phi^+{}_{mn}\end{bmatrix}e_2.$$

The determinant of A is given by $|A|=|A^-|\,e_1+|A^+|\,e_2$. Moreover, the matrix A is invertible precisely when $|A|\notin O_2$ [4,9]. The cofector of the bicomplex matrix A is given by $cof(A)=cof(A^-)e_1+cof(A^+)e_2$, where "$cof(A)$" denotes the cofector of the matrix A. The adjoint of the bicomplex matrix A is given by $adj(A)=adj(A^-)e_1+adj(A^+)e_2$, where "$adj(A)$" denotes the adjoint of the matrix A. The transpose of the bicomplex matrix A is given by $A^T=(A^-)^Te_1+(A^+)^Te_2$, where "$A^T$" denotes the transpose of the matrix A. The inverse of the bicomplex matrix A is given by $A^{-1}=(A^-)^{-1}e_1+(A^+)^{-1}e_2$. If B is a matrix with order $n\times k$, then $AB=(A^-B^-)e_1+(A^+B^+)e_2$.

Definition 1.7 (**Hermitian Matrix [8]:**). Let A be a bicomplex matrix with order $n\times n$, and if

1. A matrix A is termed i_1-Hermitian if it satisfies the condition $A=[\overline{A}]^T$.
2. A matrix A is termed i_2-Hermitian if it satisfies the condition $A=[\tilde{A}]^T$.
3. A matrix A is termed i_1i_2 -Hermitian, if it satisfies the condition $A=[A^\sharp]^T$

Theorem 1.8 *Let A be a complex matrix with order $m\times n$, then*

1. $adj(A^T)=[adj(A)]^T$.
2. $adj(A^{\theta_1})=[adj(A)]^{\theta_1}$.

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3. $\tilde{A} = A$.
4. $\overline{A} = A^\sharp$.

From the above, it is evident that $adj(A^{\theta_2}) = adj(A^T) = [adj(A)]^{\theta_2}$ *and* $adj(A^{\theta_3}) = adj(A^{\theta_1}) = [adj(A)]^{\theta_1} = [adj(A)]^{\theta_3}$.

Throughout this paper, A^T denotes the transpose of A, and A^{θ_1}, A^{θ_2}, and A^{θ_3} denote, respectively, the i_1-tranjugate, i_2-tranjugate, and i_1i_2-tranjugate of A, obtained by taking the transpose followed by the corresponding conjugation. The i_1-conjugate, i_2-conjugate, and i_1i_2-conjugate associated with a matrix A are represented by $\overline{A}$, $\tilde{A}$, and $A^\sharp$, respectively. Furthermore, $M_n(\mathbb{C}_2)$ denotes the class of all bicomplex matrices of order n, whereas $M_n(\mathbb{C}(i_1))$ represents the collection of all i_1-complex matrices of the same order.

2. Analysis of a Matrix in Bicomplex Space

This section focuses on establishing the principal results of the paper. We start by examining the relationship between transpose operations and the idempotent parts of a matrix, followed by a comprehensive analysis of matrix conjugates with respect to idempotency. We also present several important properties of the adjoint within the framework of bicomplex matrices.

Moreover, we obtain new findings related to bicomplex Hermitian matrices and study their underlying structural properties. The section further covers orthogonal bicomplex matrices along with their relevant properties and results. Altogether, these studies provide a broader and clearer understanding of matrix structures in bicomplex spaces.

Theorem 2.1. *If A is a bicomplex matrix with order $m \times n$; then the following are true in general.*

a) *The transpose of the first idempotent component of A, $(A^-)^T$, coincides with the first idempotent component of the transpose of A, $(A^T)^-$; that is, $(A^-)^T = (A^T)^-$.*
b) *The transpose of the second idempotent component of A, $(A^+)^T$, coincides with the second idempotent component of the transpose of A, $(A^T)^+$; that is, $(A^+)^T = (A^T)^+$.*
c) *The i_1-conjugate of the first idempotent component of A, $\overline{A^-}$, does not coincide with the first idempotent component of the i_1-conjugate of A, $(\overline{A})^-$; that is, $(\overline{A})^- = \overline{A^+} \neq \overline{A^-}$.*
d) *The i_1-conjugate of the second idempotent component of A, $\overline{A^+}$, does not coincide with the second idempotent component of the i_1-conjugate of A, $(\overline{A})^+$; that is, $(\overline{A})^+ = \overline{A^-} \neq \overline{A^+}$.*
e) *The i_2-conjugate of the first idempotent component of A, $\widetilde{A^-}$, does not coincide with the first idempotent component of the i_2-conjugate of A, $(\tilde{A})^-$; that is, $(\tilde{A})^- = A^+ \neq A^- = \widetilde{A^-}$.*
f) *The i_2-conjugate of the second idempotent component of A, $\widetilde{A^+}$, does not coincide with the second idempotent component of the i_2-conjugate of A, $(\tilde{A})^+$; that is, $(\tilde{A})^+ = A^- \neq A^+ = \widetilde{A^+}$.*
g) *The i_1i_2-conjugate of the first idempotent component of A, $(A^-)^\sharp$, coincides with the first idempotent component of the i_1i_2-conjugate of A, $(A^\sharp)^-$; that is, $(A^\sharp)^- = \overline{A^-} = (A^-)^\sharp$.*
h) *The i_1i_2-conjugate of the second idempotent component of A, $(A^+)^\sharp$, coincides with the second idempotent component of the i_1i_2-conjugate of A, $(A^\sharp)^+$; that is, $(A^\sharp)^+ = \overline{A^+} = (A^+)^\sharp$.*

Proof. The proof is straightforward and hence omitted.

Theorem 2.2. *Suppose A and B are matrices belonging to $M_n(\mathbb{C}_2)$. Then the adjoint of their product satisfies* $\text{adj.}(AB) = \text{adj.}(B)\text{adj.}(A)$.

Proof. Since each bicomplex matrix A and B admits an idempotent decomposition as

$$A = A^-e_1 + A^+e_2, \text{ and } B = B^-e_1 + B^+e_2.$$

Consequently,

$$AB = (A^-B^-)e_1 + (A^+B^+)e_2.$$

Therefore, by (6)

$$adj.(AB) = adj.(A^-B^-)e_1 + adj.(A^+B^+)e_2.$$

By the classical identity for complex matrices,

$$adj.(A^-B^-) = adj.(B^-)adj(A^-),\ adj.(A^+B^+) = adj.(B^+)adj.(A^+).$$

Therefore,

$$adj.(AB) = \big(adj.(B^-)adj.(A^-)\big)e_1 + \big(adj.(B^+)adj.(A^+)\big)e_2.$$

This expression coincides with

$$(adj.(B^-)e_1 + adj.(B^+)e_2)(adj.(A^-)e_1 + adj.(A^+)e_2) = adj.(B)adj.(A).$$

Hence, the theorem is proved.

Theorem 2.3. *Suppose A be a bicomplex matrix with order $m \times n$, if*

a) *A is an i_1-hermitian matrix, then $adj.(A)$ is an i_1-hermitian matrix.*
b) *A is an i_2-hermitian matrix, then $adj.(A)$ is an i_2-hermitian matrix.*
c) *A is an i_1i_2-hermitian matrix, then $adj.(A)$ is an i_1i_2-hermitian matrix.*

Proof.

1. Consider a matrix $'A'$ that is i_1-hermitian Suppose $'A'$ be an i_1-hermitian matrix, then

$$\begin{aligned} A &= A^{\theta_1} \\ \Rightarrow A^-e_1 + A^+e_2 &= (A^-)^{\theta_1}e_2 + (A^+)^{\theta_1}e_1, \text{ by (1.7)}. \\ \Rightarrow A^- = (A^+)^{\theta_1} &\text{ and } A^+ = (A^-)^{\theta_1}. \end{aligned}$$

Now

$$\begin{aligned} [adj.(A)]^{\theta_1} &= \big[adj.(A^-)e_1 + adj.(A^+)e_2\big]^{\theta_1} \\ &= \big[adj.(A^-)\big]^{\theta_1}e_2 + \big[adj.(A^+)\big]^{\theta_1}e_1, \text{ by (1.7)}. \\ &= adj.\big((A^-)^{\theta_1}\big)e_2 + adj.\big((A^+)^{\theta_1}\big)e_1, \text{ by (1.8)}. \\ &= adj.(A^+)e_2 + adj.(A^-)e_1, \text{ by (1)}. \\ &= [adj.(A)]^+e_2 + [adj.(A)]^-e_1, \text{ by (1.6)}. \\ &= adj.(A). \end{aligned}$$

2. Let $'A'$ be an i_2-hermitian matrix, then

$$\begin{aligned} A &= A^{\theta_2} \\ \Rightarrow A^-e_1 + A^+e_2 &= (A^-)^{\theta_2}e_2 + (A^+)^{\theta_2}e_1, \text{ by (1.7)}. \\ \Rightarrow A^- = (A^+)^{\theta_2} &\text{ and } A^+ = (A^-)^{\theta_2}. \end{aligned}$$

Now

$$\begin{aligned}[adj.(A)]^{\theta_2} &= [adj.(A^-)e_1 + adj.(A^+)e_2]^{\theta_2}\\ &= [adj.(A^-)]^{\theta_2} e_2 + [adj.(A^+)]^{\theta_2} e_1, \text{ by (1.7)}.\\ &= adj.((A^-)^{\theta_2})e_2 + adj.((A^+)^{\theta_2})e_1, \text{ by (1.8)}.\\ &= adj.(A^+)e_2 + adj.(A^-)e_1, \text{ by (2)}.\\ &= [adj.(A)]^+ e_2 + [adj(A)]^- e_1, \text{ by (1.6)}.\\ &= adj.(A).\end{aligned}$$

3. Let $'A'$ be an $i_1 i_2$-hermitian matrix, then

$$\begin{aligned} A &= A^{\theta_3}\\ \Rightarrow A^- e_1 + A^+ e_2 &= (A^-)^{\theta_3} e_1 + (A^+)^{\theta_3} e_2, \text{ by (1.7)}.\\ \Rightarrow A^- = (A^-)^{\theta_3} &\text{ and } A^+ = (A^+)^{\theta_3}.\end{aligned}$$

Now

$$\begin{aligned}[adj.(A)]^{\theta_3} &= [adj.(A^-)e_1 + adj.(A^+)e_2]^{\theta_3}\\ &= [adj.(A^-)]^{\theta_3} e_1 + [adj.(A^+)]^{\theta_3} e_2, \text{ by (1.7)}.\\ &= adj.((A^-)^{\theta_3})e_1 + adj.((A^+)^{\theta_3})e_2, \text{ by (1.8)}.\\ &= adj.(A^-)e_1 + adj.(A^+)e_2, \text{ by (3)}.\\ &= [adj.(A)]^- e_1 + [adj.(A)]^+ e_2, \text{ by (1.6)}.\\ &= adj.(A).\end{aligned}$$

Hence, the theorem is proved.

Definition 2.3. If A denote a matrix with order $n \times n$ whose entries are in $\mathbb{C}_2$, then A is said to be singular if its determinant satisfies

$$|A| \in O_2 = I_1 \cup I_2.$$

where

$$I_1 = \{z \cdot e_1 : z \in \mathbb{C}_1\},\ I_2 = \{w \cdot e_2 : w \in \mathbb{C}_1\}.$$

Moreover if,

1. $|A| = 0$, then A is called a strictly singular matrix.
2. $|A| \neq 0$ and $|A| \in I_1$, then A is called a non-strictly singular matrix of the first kind.
3. $|A| \neq 0$ and $|A| \in I_2$, then A is called a non-strictly singular matrix of the second kind.

Example 2.5. Let A_1, A_2, and A_3 be the bicomplex matrices such that

$$A_1 = \begin{bmatrix} e_1 & 2\\ 0 & e_2\end{bmatrix},\ A_2 = \begin{bmatrix} e_1 & 2\\ 0 & e_1\end{bmatrix},\ \text{and } A_3 = \begin{bmatrix} e_2 & 2\\ 0 & e_2\end{bmatrix}.$$

Then$|A_1| = 0$, $|A_2| = e_1$, and $|A_3| = e_2$. Thus A_1, A_2, and A_3 are the strictly singular matrix, non-strictly singular matrix of the first kind, and non-strictly singular matrix of the second kind, respectively.

The classification into strictly singular and non-strictly singular matrices clarifies the role of idempotent components e_1 and e_2 in determinant degeneracy, providing insight for bicomplex linear operators and their spectral analysis.

Theorem 2.6. *Let A denote a bicomplex valued matrix with order $n \times n$ such that $|A| \in O_2$, then $|adj.(A)| \in O_2$.*

Proof. Since

$$\begin{aligned}|adj.(A)| &= |adj.(A^-)|\, e_1 + |adj.(A^+)|\, e_2, \text{ by (1.6)}.\\ &= |A^-|^{(n-1)} e_1 + |A^+|^{(n-1)} e_2.\end{aligned}$$

Since $|A| \in O_2$, therefore $|A^-| = 0$ or $|A^+| = 0$. Thus, by (4), $|adj.(A)| \in O_2$.

Hence, the theorem is proved.

Theorem 2.7. *Let A denote a bicomplex valued matrix with order $n \times n$ such that A is a strictly singular matrix, then $adj.(A)$ is a strictly singular matrix.*

Proof. Since

$$\begin{aligned}|adj.(A)| &= |adj.(A^-)|\, e_1 + |adj.(A^+)|\, e_2, \text{ by (1.6)}.\\ &= |A^-|^{(n-1)} e_1 + |A^+|^{(n-1)} e_2.\end{aligned}$$

Since $|A| = 0$, therefore $|A^-| = 0$ and $|A^+| = 0$. Thus, by (5), $|adj.(A)| = 0$.

Hence, the theorem is proved.

Theorem 2.8. *Consider two $n \times n$ bicomplex matrices A and B. If their product satisfies $A \cdot B = 0$, then A or B are singular matrix; equivalently $|A| \in O_2$ or $|B| \in O_2$.*

Proof. Since, product of matrices A and B is zero matrix, that is;

$$\begin{aligned} 0 &= (A)(B)\\ &= (A^- B^-)e_1 + (A^+ B^+)e_2, \text{ by (1.6)}.\\ &\Rightarrow (A^-)(B^-) = 0,\ (A^+)(B^+) = 0\\ &\Rightarrow |(A^-)(B^-)| = 0,\ |(A^+)(B^+|) = 0\\ &\Rightarrow |(A^-)|\,|(B^-)| = 0,\ |(A^+)|\,|(B^+)| = 0\end{aligned}$$

Therefore, $|(A^-)| = 0$ or $|(B^-)| = 0$. thus, A is a singular matrix or B is a singular matrix.

Hence, the theorem is proved.

Theorem 2.9. *If A, B are two matrices with bicomplex entriess with order $n \times n$ such that $A \cdot B = 0$ and $|B| \neq 0$, then $|A| \in O_2$.*

Proof. Since, given product is zero matrix, that is

$$\begin{aligned} 0 &= AB\\ &= (A^- B^-)e_1 + (A^+ B^+)e_2, \text{ by (1.6)}\\ &\Rightarrow (A^-)(B^-) = 0,\ (A^+)(B^+) = 0\\ &\Rightarrow |(A^-)(B^-)| = 0,\ |(A^+)(B^+)| = 0\\ &\Rightarrow |(A^-)|\,|(B^-)| = 0,\ |(A^+)|\,|(B^+)| = 0\end{aligned}$$

Since $|B| \neq 0$, that is, $|B^-| \neq 0$ or $|B^+| \neq 0$. By (6), $|A^-| = 0$ or $|A^+| = 0$. Thus$|A| \in O_2$.

Hence, the theorem is proved.

Theorem 2.10. *If A,B are two matrices with bicomplex entries of order $n \times n$ satisfyimg $A \cdot B = 0$ and $|B| \notin O_2$, then A must be zero matrix.*
Proof. Since

$$\begin{aligned} 0 &= A \cdot B \\ &= (A^- \cdot B^-)e_1 + (A^+ \cdot B^+)e_2, \text{ by (1.6)}. \\ \Rightarrow A^- \cdot B^- = 0 &\quad \text{and} \quad A^+ \cdot B^+ = 0. \end{aligned}$$

Since $|B| \notin O_2$, that is, B is not a singular matrix, so $|B^-| \neq 0$ and $|B^+| \neq 0$. Thus $(B^-)^{-1}$ and $(B^+)^{-1}$ exist. By (7), we get

$$A^- = 0 \text{ and } A^+ = 0$$
$$\Rightarrow \quad A = 0.$$

Hence, the theorem is proved. □
Corollaries (2.11) and (2.12) follow directly from the result established in (2.9) and (2.10).
Corollary 2.11. *If A,B are two matrices with bicomplex entries with order $n \times n$ such that $A \cdot B = 0$ and $|A| \neq 0$, then $|B| \in O_2$.*
Corollary 2.12. *If A,B are two matrices with bicomplex entries with order $n \times n$ such that $A \cdot B = 0$ and $|A| \notin O_2$, then $B = 0$.*
Theorem 2.13. *If A,B are two matrices with bicomplex entries of order $n \times n$ satisfyimg $A \cdot B = 0$ and $A \neq 0$ and $B \neq 0$, then both A , B are singular matrices; equivalently $|A| \in O_2$ and $|B| \in O_2$.*
Proof. Since

$$\begin{aligned} 0 &= A \cdot B \\ &= (A^- \cdot B^-)e_1 + (A^+ \cdot B^+)e_2, \text{ by (1.6)}. \\ &\Rightarrow A^- \cdot B^- = 0\ ,\ A^+ \cdot B^+ = 0. \end{aligned}$$

Now, we assume to the contrary that A is not a singular matrix, that is, $|A^-| \neq 0$ and $|A^+| \neq 0$. Thus $(A^-)^{-1}$ and $(A^+)^{-1}$ exist. By (8), we get

$$B^- = 0 \text{ and } B^+ = 0$$
$$\Rightarrow \quad B = 0.$$

In the same manner, if we assume that B is not a singular matrix, then we get

$$A = 0.$$

Thus, (9) and (10) show a contradiction. Therefore, our assumption that A is not a singular matrix and B is not a singular matrix is not true. Hence both A and B are singular matrices.
Hence, the theorem is proved. □
Definition 2.14. Let A be a matrices with bicomplex entries with order $n \times n$, then A is called an orthogonal matrix if it satisfies

$$A^T A = I = AA^T,$$
$$\text{i.e.,} \qquad A^T = A^{-1}.$$

It follows directly that whenever a bicomplex matrix A satisfies the orthogonality condition, both its transpose A^T and its inverse A^{-1} also preserve this property and hence are orthogonal bicomplex matrices.
Example 2.15. Let A be a bicomplex matrix such that $A = \begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}$, then $A^T = \begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix} = A$ and $AA^T = A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus A is an orthogonal bicomplex matrix.
Theorem 2.16. *Let 'A' be an orthogonalmatrices with bicomplex entries with order $n \times n$ if and only if A^- and A^+ are orthogonal complex matrices.*
Proof. Let $'A'$ be an orthogonal bicomplex matrix with order $n \times n$, that is,

$$\begin{aligned} & A^{-1} = A^T. \\ \Longleftrightarrow\ & (A^- e_1 + A^+ e_2)^{-1} = (A^- e_1 + A^+ e_2)^T. \\ \Longleftrightarrow\ & (A^-)^{-1} e_1 + (A^+)^{-1} e_2 = (A^-)^T e_1 + (A^+)^T e_2, \text{ by (1.6)}. \\ \Longleftrightarrow\ & (A^-)^{-1} = (A^-)^T \text{and } (A^+)^{-1} = (A^+)^T. \end{aligned}$$

Hence, the theorem is proved. □
Theorem 2.17. *If A is an orthogonal matrices with bicomplex entries with order $n \times n$, then $|A| \in \{1, -1, i_1 i_2, -i_1 i_2\}$.*
Proof. Let $'A'$ be an orthogonal bicomplex matrix with order $n \times n$, then

$$\begin{aligned} & A^T A = I = AA^T \\ \Rightarrow\ & |A||A| = 1, \text{ since } |A| = |A^T| \\ \Rightarrow\ & |A^- e_1 + A^+ e_2||A^- e_1 + A^+ e_2| = 1 \\ \Rightarrow\ & (|A^-|e_1 + |A^+|e_2)(|A^-|e_1 + |A^+|e_2) = 1 \\ \Rightarrow\ & (|A^-|)^2 e_1 + (|A^+|)^2 e_2 = e_1 + e_2 \\ \Rightarrow\ & |A^-|^2 = 1 \text{ and } |A^+|^2 = 1 \\ \Rightarrow\ & |A^-| = \pm 1 \text{ and } |A^+| = \pm 1 \end{aligned}$$

Thus, the possible values of $|A|$ are $1, -1, e_1 - e_2$, and $e_2 - e_1$ since $e_1 - e_2 = i_1 i_2$.
Hence, the theorem is proved. □
Definition 2.18 Let $'A'$ be an orthogonal bicomplex matrix with order $n \times n$ and if
1. $|A| =$ 1 or -1, then A is called a trivial orthogonal bicomplex matrix.
2. $|A| = i_1 i_2$ or $-i_1 i_2$, then A is called a non-trivial orthogonal bicomplex matrix.

Example 2.19. Consider, B are two matrices with bicomplex entries like as $A = \begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix}$, and $B = \begin{bmatrix} -e_1 & -e_2 \\ e_2 & -e_1 \end{bmatrix}$, then

$$A^T = \begin{bmatrix} e_1 & e_2 \\ e_2 & e_1 \end{bmatrix} = A,\ AA^T = A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};$$
$$B^T = \begin{bmatrix} -e_1 & e_2 \\ -e_2 & -e_1 \end{bmatrix},\ BB^T = B^T B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $|A| = i_1 i_2$ and $|B| = 1$. Thus A and B are non-trivial and trivial orthogonal bicomplex matrices, respectively.
Theorem 2.20. *Let A, B be two bicomplex orthogonal matrices of order $n \times n$, then the product AB is an orthogonal bicomplex matrix, that is, the class of orthogonal bicomplex matrices is closed under multiplication.*
Proof. Since A and B are orthogonal bicomplex matrices, therefore

$$A^T = A^{-1} \text{ and } B^T = B^{-1}.$$
$$\Rightarrow (A^-)^T e_1 + (A^+)^T e_2 = (A^-)^{-1} e_1 + (A^+)^{-1} e_2 \text{ and } (B^-)^T e_1 + (B^+)^T e_2 = (B^-)^{-1} e_1 + (B^+)^{-1} e_2.$$
$$\Rightarrow (A^-)^T = (A^-)^{-1},\ (A^+)^T = (A^+)^{-1},\ (B^-)^T = (B^-)^{-1},\ (B^+)^T = (B^+)^{-1}.$$

Now

$$
\begin{aligned}
(AB)^{-1} &= ((AB)^{-})^{-1}e_1 + ((AB)^{+})^{-1}e_2, \text{ by (1.6).} \\
&= (A^{-}B^{-})^{-1}e_1 + (A^{+}B^{+})^{-1}e_2, \text{ by (1.6).} \\
&= (B^{-})^{-1}(A^{-})^{-1}e_1 + (B^{+})^{-1}(A^{+})^{-1}e_2 \\
&= (B^{-})^{T}(A^{-})^{T}e_1 + (B^{+})^{T}(A^{+})^{T}e_2, \text{ by (11).} \\
&= (A^{-}B^{-})^{T}e_1 + (A^{+}B^{+})^{T}e_2. \\
&= ((AB)^{-})^{T}e_1 + ((AB)^{+})^{T}e_2, \text{ by (1.6).} \\
&= (AB)^{T}.
\end{aligned}
$$

Hence, the theorem is proved.

Conclusion

In this paper, we studied algebraic properties of bicomplex matrices via their idempotent decomposition. It was shown that fundamental operations such as transpose, cofactor, determinant, and adjoint behave consistently with their corresponding idempotent components.

The notions of singular and strictly singular bicomplex matrices were introduced and characterized. Key results involving adjoint matrices, including $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$ and preservation of Hermitian structure, were established. Further, several properties related to products of bicomplex matrices were obtained, particularly conditions linking zero products with singularity.

We also introduced orthogonal bicomplex matrices and showed that their determinants lie in $\{1, -1, i_1 i_2, -i_1 i_2\}$ and that this class is closed under multiplication. These results provide a concise framework for understanding the structure of bicomplex matrices and support further developments in bicomplex linear algebra.

References

1. Daniel Alpay, Izchak Lewkowicz, and Mihaela Vajiac, Interpolation with symmetry and a herglotz theorem in the bicomplex setting, Journal of Mathematical Analysis and Applications 524 (2023), no. 2, 127201.
2. Anjali and Amita, Certain results on bicomplex matrices, Global Journal of Science Frontier Research 18 (2018), no. F2, 7–26.
3. Anjali, Akhil Prakash, Amita, and Neeraj Kumar Tomar, Idempotent & nilpotent operators and matrices in bicomplex space, THE MATHEMATICS EDUCATION LIX (2025), no. 2, June 2025.
4. Anjali, Fahed Zulfeqarr, Akhil Prakash, and Prabhat Kumar, Matrix representations of linear transformations on bicomplex space, Asian-European Journal of Mathematics 17 (2024), no. 11, 2450079.
5. John B Fraleigh, *A first course in abstract algebra*, Pearson Education India, 2003.
6. Pertti Lounesto, *Clifford algebras and spinors*, 2 ed., Cambridge University Press, Cambridge, 2001.
7. M Elena Luna-Elizarrarás, Michael Shapiro, Daniele C Struppa, and Adrian Vajiac, *Bicomplex holomorphic functions: the algebra, geometry and analysis of bicomplex numbers*, Birkhäuser, 2015.
8. Akhil Prakash and Prabhat Kumar, *Certain study of bicomplex matrices and a new composition of bicomplex matrices*, Global Journal of Science Frontier Research 18 (2018), no. F3, 55–78.
9. G Baley Price, *An introduction to multicomplex spaces and functions*, CRC Press, 2018.
10. Dominic Rochon and Michael Shapiro, *On algebraic properties of bicomplex and hyperbolic numbers*, Anal. Univ. Oradea, fasc. math 11 (2004), no. 71, 110.
11. Stefan Rönn, *Bicomplex algebra and function theory*, arXiv preprint math/0101200 (2001).
12. Corrado Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici*, Mathematische Annalen 40 (1892), no. 3, 413–467.
13. Rajiv K Srivastava and Jogendra Kumar, *A note on zeros of the bicomplex riemann zeta function*, Int. J. of Math. Sci. and Engg. Appl.(IJMSEA) 2 (2008), 163–172.