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Second Order Fuzzy Relational Equations with $Sup-i$ and $Inf-w_i$ Compositions: A Theoretical and Computational Framework

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Abstract

Second-order fuzzy relations (SOFRs), where membership grades are themselves fuzzy numbers, can model multiple levels of uncertainty. Max-min, max-product, and sum-product have already been established within the efficient epsilon-delta ($\varepsilon-\delta$) formalism, however the exploration of more general fuzzy relational connectives remains limited. This paper introduces two non-standard compositions, the Supremum-t $-$norm (Sup$-$i) and the Infimum-s $-$conorm (Inf$-w_i$) compositions, within a comprehensive $\varepsilon-\delta$ framework for formulating and solving Second Order Fuzzy Relational Equations (SOFREs). We extend these compositions founded by Klir & Yuan [1] for Type-1 relations to the second-order context. We prove key algebraic properties, including a necessary condition for solvability, and demonstrate the application of these compositions through a comparative numerical example in a medical diagnostic setting. Our results demonstrate that the Sup$-$i composition offers a tunable, pessimistic chaining of relations, as shown with the product t $-$norm. In contrast, the Inf$-w_i$ composition provides an optimistic, evidence-aggregating behavior when implemented with probabilistic sum s $-$conorm; it yields strong diagnostic discrimination but at the cost of significant uncertainty amplification. Together, these compositions significantly expand the modeling flexibility of the second-order fuzzy relational calculus.

Keywords: Second Order Fuzzy Relations, Second-Order Fuzzy Relational Equations, Epsilon-Delta Fuzzy Numbers, Sup$-$i Composition, Inf$-w_i$ Composition, t $-$norms, s $-$conorms.

1. Introduction

Fuzzy Relational Equations (FREs) based on max-min composition were first proposed and investigated by Sanchez [2]. FREs are foundational frameworks in fuzzy set theory that model uncertain relations between variables [1, 3]. The max-min composition is appropriate due to its algebraic properties and direct semantic interpretation as a "strongest path" [4, 5].The limitation of Type-1 fuzzy sets in handling uncertainty about the membership degrees themselves led to the development of higher-order fuzzy sets [6, 9]. Second-order fuzzy relations, where membership grades are fuzzy numbers, directly address this limitation [7, 8]. However, specialized, efficient formalisms are required to reduce the computational complexity of general Type-2 fuzzy systems. Klir & Yuan [1] systematically established compositions based on general triangular norms (t $-$norms) and conorms (s $-$conorms) for FREs. The $Sup-i$ composition generalizes the max-min and max-product compositions, and the dual $Inf-w_i$ composition generalizes the min-max and min-sum. These generalized compositions offer a more flexible framework of logical connectives or aggregation methods, allowing for more specific and nuanced modeling of how relations chain together. Analytical methods for solving Type-1 fuzzy relational equations with max-product composition were established by Peeva and Kyosev. These methods provide universal algorithm for computing the greatest solution and the set of all minimal solutions, when the system is consistent [11]. Li and Fang provides a thorough investigation into solving a finite system of fuzzy relational equations using Sup$-T$ composition, where T is continuous triangular norm [12].A significant gap exists in the compositions of second order fuzzy relations due to its computational complexity. This paper bridges this gap by introducing and formalizing the $Sup-i$ and $Inf-w_i$ compositions for second order fuzzy relations in $\varepsilon-\delta$ framework. This work extends the composition set explored in [8, 14] by providing: (1) a generalized composition framework supporting arbitrary t $-$norms and s $-$conorms, (2) a complete mathematical foundation for approximate fuzzy operations that maintain conservative approximation properties, and (3) the first characterization of uncertainty propagation patterns across different composition types. These contributions established a comprehensive framework for second order fuzzy relational algebra.

Our primary contributions are:

1.The first formal definition of $Sup-i$ and $Inf-w_i$ compositions for second order fuzzy relations within the $\varepsilon-\delta$ formalism.

2.The development of complete computational procedures for these compositions, extending the approximate arithmetic for $\varepsilon-\delta$ numbers.

3.An analysis of key algebraic properties, including a discussion on solvability conditions for the corresponding SOFREs.

4.A quantitatively verified numerical demonstration, including an analysis of uncertainty propagation patterns and clinical pathways, comparing the behavior of $Sup-i$ (with product and minimum t $-$norms) and $Inf-w_i$ (with probabilistic sum s $-$conorm) against the standard max-min composition.

This work provides a more flexible and expressive framework for modeling systems under higher-order uncertainty.

2. Preliminaries

This section presents some definitions and preliminaries related to compositions and fuzzy relations.

2.1. Epsilon-Delta ($\varepsilon-\delta$) Triangular Fuzzy Numbers [8]

If r is a real number, then an $\varepsilon-\delta$ fuzzy number $r_{\varepsilon,\delta}$ is the triangular fuzzy number, for some $\varepsilon,\delta\in\mathbb{R},(\varepsilon,\delta>0)$is a fuzzy set $r_{\varepsilon,\delta}:\mathbb{R}\to[0,1]$defined by

$$r_{\varepsilon,\delta}(x)=\begin{cases}\frac{x-(r-\varepsilon)}{\varepsilon}, \text{if } r-\varepsilon<x\le r,\\ \frac{x-(r+\delta)}{-\delta}, \text{if } r\le x<r+\delta,\\ 0, otherwise.\end{cases}$$

Also,$r_{\varepsilon,\delta}=(r-\varepsilon,r,r+\delta)$.

2.2. Second Order Fuzzy Relation [8]

A function $R^2:X\times Y\to\mathbb{F}(I)$ defined by
$R^2(X\times Y)=r_{\varepsilon,\delta}^{xy}$, where

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$r^{xy}_{\varepsilon,\delta} = r_{\varepsilon,\delta}$is a triangular fuzzy number on $I = [0,1]$ such that $r - \varepsilon \geq 0, r + \delta \leq 1$, which indicates the fuzzy relationship between x and y for every $x \in X, y \in Y, \mathbb{F}(I)$is a fuzzy power set of $I = [0,1]$, is called a second order fuzzy relation. It is denoted by$R^2(X \times Y) = r^{xy}_{\varepsilon,\delta}$.

2.3. Approximate Min and Max Operations on Epsilon-Delta Fuzzy Numbers

Let $r_{\varepsilon_1,\delta_1}$and $s_{\varepsilon_2,\delta_2}$ be any two epsilon-delta fuzzy numbers where $r \leq s$.

1. If $r - \varepsilon_1 \leq s - \varepsilon_2$ and $r + \delta_1 \leq s + \delta_2$, then:
$\min(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2} = r_{\varepsilon_1,\delta_1}$ and
$\max(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2} = s_{\varepsilon_2,\delta_2}$.
2. If $s - \varepsilon_2 < r - \varepsilon_1$ and $r + \delta_1 \leq s + \delta_2$, then:
$\min(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2} = r_{\varepsilon_2-|r-s|,\delta_1}$ and
$\max(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2} = s_{\varepsilon_1+|r-s|,\delta_2}$.
3. If $r - \varepsilon_1 \leq s - \varepsilon_2$ and $s + \delta_2 < r + \delta_1$,then:
$\min(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2} = r_{\varepsilon_1,\delta_2+|r-s|}$ and
$\max(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2} = s_{\varepsilon_2,\delta_1-|r-s|}$.
4. If $s - \varepsilon_2 < r - \varepsilon_1$ and $s + \delta_2 < r + \delta_1$, then:
$\min(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \wedge s_{\varepsilon_2,\delta_2} = r_{\varepsilon_2-|r-s|,\delta_2+|r-s|}$ and
$\max(r_{\varepsilon_1,\delta_1}, s_{\varepsilon_2,\delta_2}) = r_{\varepsilon_1,\delta_1} \vee s_{\varepsilon_2,\delta_2} = s_{\varepsilon_1+|r-s|,\delta_1-|r-s|}$.

These operations provide triangular approximations of supremum and infimum operations on fuzzy numbers **[8].**

2.4. Approximate Operations on $\varepsilon - \delta$ Numbers [8]

Let $A = r_{\varepsilon_1,\delta_1}$and $B = s_{\varepsilon_2,\delta_2}$ be two $\varepsilon - \delta$ fuzzy numbers where $r \leq s$. The piecewise definitions for the approximate minimum ($\wedge$) and maximum ($\vee$) operations are as defined in 2.3. These operations provide triangular approximations of Zadeh's extension principle for supremum and infimum.

The product of two $\varepsilon - \delta$ numbers is approximated by [6]:

$$A.B \approx (r.s)_{r\varepsilon_2+s\varepsilon_1-\varepsilon_1\varepsilon_2, r\delta_2+s\delta_1+\delta_1\delta_2}$$

2.5. Type-1 *Sup*$-i$ and *Inf*$-w_i$ Compositions [1]

Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be Type-1 fuzzy relations. Let i be a t −norm and u be an s-conorm.

The ***Sup*$-i$ composition** $S \circ_i R$ is defined by:

$$\mu_{S\circ_i R}(x,z) = \sup_{y\in Y}[\mu_R(x,y) i \mu_S(y,z)]$$

This generalizes max-min (i =min) and max-product (i =product).

The ***Inf*$-w_i$ composition** $S \circ^u R$ is defined by:

$$\mu_{S\circ^u R}(x,z) = \inf_{y\in Y}[\mu_R(x,y) u \mu_S(y,z)]$$

This is the dual composition of *Sup*$-i$.

3. Second Order Fuzzy Relational Equations with *Sup*$-i$ and *Inf*$-w_i$ Compositions

The *Sup*$-i$ and *Inf*$-w_i$ compositions are now extended to the second-order context.

3.1. Compositions of Second-Order Fuzzy Relations

Let $R^2 \subseteq X \times Y$ and $S^2 \subseteq Y \times Z$ be two second-order fuzzy relations with membership function given by,
$\mu_{R^2}(x,y) = r^{xy}_{\varepsilon_{xy},\delta_{xy}}$ and $\mu_{S^2}(y,z) = s^{yz}_{\phi_{yz},\psi_{yz}}$.

3.1.1. *Sup*$-i$ Composition

The *Sup*$-i$ composition $S^2 \circ_i R^2$ is a second-order fuzzy relation on $X \times Z$ defined by,

$$\mu_{S^2\circ_i R^2}(x,z) = \bigvee_{y\in Y}[\mu_{R^2}(x,y)\hat{\imath}\mu_{S^2}(y,z)]$$

where $\vee$ is the approximate supremum operation for $\varepsilon - \delta$ numbers, and $\hat{\imath}$ is an approximate t-norm operation defined for $\varepsilon - \delta$ numbers.

3.1.2. *Inf*$-w_i$ Composition

The *Inf*$-w_i$ composition $S^2 \circ^u R^2$ is a second-order fuzzy relation on $X \times Z$ defined by,

$$\mu_{S^2\circ^u R^2}(x,z) = \bigwedge_{y\in Y}[\mu_{R^2}(x,y)\hat{u}\mu_{S^2}(y,z)]$$

where $\wedge$ is the approximate infimum operation for $\varepsilon - \delta$ numbers, and $\hat{u}$ is an approximate s −conorm operation defined for $\varepsilon - \delta$ numbers.

3.2. Computational Formulation

Defining the approximate operations $\hat{\imath}$ and $\hat{u}$ is the computational challenge. The proposed definitions are based on the interval arithmetic of the α-cuts of the $\varepsilon - \delta$ fuzzy numbers.

3.2.1. Approximate t −norm ($\hat{\imath}$)

Let $A = r_{\varepsilon_1,\delta_1}$and $B = s_{\varepsilon_2,\delta_2}$ be two $\varepsilon - \delta$ fuzzy numbers, and i be a continuous t −norm. The approximate t −norm operation $A\hat{\imath}B$ produces a new ε −δ fuzzy number $C = t_{\varepsilon_3,\delta_3}$ where the core value is $t = i(r,s)$. The spreads ε_3, δ_3 are computed such that α −cut $C[\alpha]$ contains the interval resulting from the t −norm applied to the α −cuts of A and$B$:

$$C[\alpha] \supseteq \{i(a,b) | a \in A[\alpha] \text{ and } b \in B[\alpha]\}$$

For specific t −norms, this leads to the following closed-form formulas:

For the Minimum *t*-norm ($i(a,b) = min(a,b)$): The result is given directly by the existing approximate minimum operation $\wedge$, denoted here as $A\hat{\imath}_{min}B$.

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For the Product t −norm ($i(a,b) = a \times b$): The approximation is derived from the product of the α −cut intervals.

$$A\,\hat{\imath}_{prod}\,B \approx t_{\varepsilon_3,\delta_3}$$

where:

$$t = r \times s$$
$$\varepsilon_3 = r \times \varepsilon_2 + s \times \varepsilon_1 - \varepsilon_1 \times \varepsilon_2$$
$$\delta_3 = r \times \delta_2 + s \times \delta_1 + \delta_1 \times \delta_2$$

3.3 Approximate s −conorm $\hat{u}$

Let $A = r_{\varepsilon_1,\delta_1}$ and $B = s_{\varepsilon_2,\delta_2}$, be two $\varepsilon - \delta$ fuzzy numbers, and let u be a continuous s −conorm. The approximate s −conorm operation $A\,\hat{u}\,B$ produces a new $\varepsilon - \delta$ fuzzy number $C = t_{\varepsilon_3,\delta_3}$, where the core value is $t = u(r,s)$. The spreads ε_3 and δ_3 are computed to satisfy:

$$C[\alpha] \supseteq \{\mathrm{u(a,b)}\ |a \in A[\alpha] \text{ and } b \in B[\alpha]\}$$

For the Probabilistic Sum s −conorm ($u(a,b) = a + b - a \times b$):
The approximation results from the addition and product of the α −cut intervals.

$$A \circ^u B \approx t_{\varepsilon_3,\delta_3}$$

where:

$$t = r + s - r \times s$$
$$\varepsilon_3 = \varepsilon_1 + \varepsilon_2 + r \times \delta_2 + s \times \delta_1 + \delta_1 \times \delta_2$$
$$\delta_3 = \delta_1 + \delta_2 + r \times \varepsilon_2 + s \times \varepsilon_1 - \varepsilon_1 \times \varepsilon_2$$

The resultant fuzzy number $t_{\varepsilon_3,\delta_3}$ may lie outside the interval 0 and 1. To keep a valid fuzzy number within [0, 1] and preserve the containment property, we treat this as a constrained optimization. The resulting fuzzy number $t_{\varepsilon_3,\delta_3}$ must satisfy:

1. $\boldsymbol{C[\alpha] = [t - (1-\alpha)\varepsilon_3, t + (1-\alpha)\delta_3]}$ α-Cut Representation
2. $\boldsymbol{C[\alpha] \supseteq \{ u(a,b) \mid a \in A[\alpha], b \in B[\alpha] \}}$ (Containment)
3. $\boldsymbol{0 \leq t - \varepsilon_3 \leq t \leq t + \delta_3 \leq 1}$ (Validity)

The practical implementation applies constrained bounds:

$\boldsymbol{t_{final} = min(t, 1)}$

$\boldsymbol{\varepsilon_{3final} = min(\varepsilon_3, t_{final})}$ (Ensures lower bound $\geq$ 0)

$\boldsymbol{\delta_{3final} = min(\delta_3, 1 - t_{final})}$ (ensures upper bound$\leq$ 1)

This method ensures that the result is mathematically correct, but be aware that there may be a slight change of precision when applying various composition operations.

The definitions of $\hat{\imath}$ and $\hat{u}$ ensure that resultant $\varepsilon - \delta$ fuzzy number C contains all possible values that resulted from applying Zadeh's extension principle to the continuous t −norm i [7].

3.4. Solvability of SOFREs

The fundamental SOFRE has the form:

$$R^2 = P^2 \circ Q^2$$

where one of the three second order relations P^2, Q^2,or R^2 is need to be determined [2,3].

3.4.1 Theorem (Necessary condition for Solvability with *Sup*−i Composition)

Let $R^2 = P^2 \circ_i Q^2$ be a consistent SOFRE. A necessary condition for the existence of a solution for the unknown relation Q^2 (with P^2 and R^2 given) is that for every$(x,z) \in X \times Z$, there exists at least one $y \in Y$ such that:

$$r^{xz} \leq p^{xy}\ i\ q^{yz}$$

where r^{xz},p^{xy}, q^{yz} are the core values of the respective $\varepsilon - \delta$ numbers. A similar condition, based on the inequality $r^{xz} \geq p^{xy}\ u\ q^{yz}$, holds for the *Inf*−$w_i$ composition.

Proof: The proof follows from the definition of the compositions and the properties of t −norms and s −conorms. For the *Sup*−i composition, the resultant is the supremum over Y for each (x,z) pair. If for a given(x,z), the core of $R^2(x,z)$ is greater than all possible $p^{xy}\ i\ q^{yz}$, no solution exists. The condition is a direct translation of this logic to the second order core values.

Discussion on Solvability:

The condition in 3.4.1 Theorem is necessary but not sufficient. A complete solution to a SOFRE requires the core values q^{yz} for unknown relation Q^2 and associated uncertainty spreads$(\varepsilon_{yz}, \delta_{yz})$. The propagation of these uncertainties through the approximate operations $\hat{\imath}$ or $\hat{u}$ and the subsequent approximate supremum/infimum presents a significantly more complex challenge than in the Type-1 case.

Establishing necessary and sufficient conditions for solvability, or developing practical algorithms to solve for the full parameter set of Q^2, remains an open and non-trivial research problem [2,13]. The successful implementation and comparative analysis of these compositions in Section 4 validates this foundational framework and provides a concrete basis for future work on the solvability problem.

3.5. Analysis of Key Algebraic Properties

Beyond solvability, it is important to characterize other fundamental algebraic properties of the proposed compositions to understand their behavior in relational calculus. The following properties are analyzed in the context of second order fuzzy relations.

3.5.1 Monotonicity Property

The *Sup*−i and *Inf*−$w_i$compositions are monotonic with respect to the fuzzy relation order. That is, for second order relations P^2, Q^2, R^2, S^2 if $P^2 \subseteq Q^2$ and $R^2 \subseteq S^2$, then:

$$P^2 \circ_i R^2 \subseteq Q^2 \circ_i S^2$$

and

$$P^2 \circ^u R^2 \subseteq Q^2 \circ^u S^2$$

where the pointwise fuzzy subset relation for $\varepsilon - \delta$ numbers defined the order $\subseteq$.

Proof:

Part 1: Monotonicity of *Sup−i* **Composition**
Let $P^2 \subseteq Q^2$ and $R^2 \subseteq S^2$. We need to show that for all $(x,z) \in X \times Z$:

$$\mu_{P^2 \circ_i R^2}(x,z) \subseteq \mu_{Q^2 \circ_i S^2}(x,z)$$

By definition,

$$\mu_{P^2 \circ_i R^2}(x,z) = \bigvee_{y \in Y}[\mu_{P^2}(x,y) \hat{\imath} \mu_{R^2}(y,z)]$$

$$\mu_{Q^2 \circ_i S^2}(x,z) = \bigvee_{y \in Y}[\mu_{Q^2}(x,y) \hat{\imath} \mu_{S^2}(y,z)]$$

Since $P^2 \subseteq Q^2$ and $R^2 \subseteq S^2$, we have for all x, y, z:
$\mu_{P^2}(x,y) \subseteq \mu_{Q^2}(x,y)$ and $\mu_{R^2}(y,z) \subseteq \mu_{S^2}(y,z)$
The approximate t −norm operation $\hat{\imath}$ is monotonic by construction (as it operates on α −cut intervals and preserves inclusion). Therefore:
$\mu_{P^2}(x,y) \hat{\imath} \mu_{R^2}(y,z) \subseteq \mu_{Q^2}(x,y) \hat{\imath} \mu_{S^2}(y,z) \qquad \forall y \in Y$
The approximate supremum operation $\vee$ over all $y \in Y$ preserves this inclusion relationship, as it selects the "largest" fuzzy number from the set of pairwise results. Thus:

$$\bigvee_{y \in Y}[\mu_{P^2}(x,y) \hat{\imath} \mu_{R^2}(y,z)] \subseteq \bigvee_{y \in Y}[\mu_{Q^2}(x,y) \hat{\imath} \mu_{S^2}(y,z)]$$

Hence, $P^2 \circ_i R^2 \subseteq Q^2 \circ_i S^2$

Part 2: Monotonicity of *Inf−w*$_i$ **Composition**
The proof follows similarly. For all $(x,z) \in X \times Z$:

$$\mu_{P^2 \circ^u R^2}(x,z) = \bigwedge_{y \in Y}[\mu_{P^2}(x,y) \hat{u} \mu_{R^2}(y,z)]$$

$$\mu_{Q^2 \circ^u S^2}(x,z) = \bigwedge_{y \in Y}[\mu_{Q^2}(x,y) \hat{u} \mu_{S^2}(y,z)]$$

Given the same inclusion relationships and the monotonicity of $\hat{u}$, we have:
$\mu_{P^2}(x,y) \hat{u} \mu_{R^2}(y,z) \subseteq \mu_{Q^2}(x,y) \hat{u} \mu_{S^2}(y,z) \qquad \forall y \in Y$
The approximate infimum operation $\wedge$ preserves this inclusion relationship, yielding:

$$\bigwedge_{y \in Y}[\mu_{P^2}(x,y) \hat{u} \mu_{R^2}(y,z)] \subseteq \bigwedge_{y \in Y}[\mu_{Q^2}(x,y) \hat{u} \mu_{S^2}(y,z)]$$

Therefore,

$$P^2 \circ^u R^2 \subseteq Q^2 \circ^u S^2$$

This completes the proof of monotonicity for both compositions

3.5.2 Non-Associativity property

Unlike the specific case of the max-min composition [8], the *Sup−i* composition is generally not associative for a general t −norm i. That is, for general second order relations,

$$(P^2 \circ_i Q^2) \circ_i R^2 \neq P^2 \circ_i (Q^2 \circ_i R^2)$$

Similarly, the *Inf−w*$_i$ composition is generally not associative for a general s −conorm u.
Justification: The lack of associativity is a known characteristic of sup−t compositions in general Type-1 fuzzy logic [1, 4, 5]. This property carries over to the second order framework due to the interplay between the supremum and the t −norm operations. The failure of associativity implies that the order of composition in multi-stage reasoning chains must be carefully considered.

3.5.3. Relationship with Standard Compositions

The max-min and max-product compositions are specific instances of the more general framework presented here. Specifically, the max-min composition is equivalent to the *Sup−i* composition where i is the minimum t −norm and the max-product composition is equivalent to the *Sup−i* composition where i is the product t −norm

4. Numerical Example and Comparative Analysis

We demonstrate the new compositions using a simplified medical diagnosis scenario adapted from [2, 9].

4.1 Setup:

Sets: $X = \{P_1, P_2\}$(Two patients with different symptom profiles),
$Y = \{Fever, Cough, Fatigue\}$ (Three symptoms),
$Z = \{CommonCold, Influenza, Pneumonia\}$ (Three diseases).

Patient-Symptom Relation (P^2): This relation captures the fuzzy membership of each patient to each symptom, represented as $\varepsilon - \delta$ triplets.
$\mu_{P_1^2}(P_1, \text{Fever}) = 0.85_{0.1,0.07}$, $\mu_{P_1^2}(P_1, \text{Cough}) = 0.5_{0.15,0.15}$,
$\mu_{P_1^2}(P_1, \text{Fatigue}) = 0.90_{0.1,0.05}$.
$\mu_{P_2^2}(P_2, \text{Fever}) = 0.6_{0.1,0.15}$, $\mu_{P_2^2}(P_2, \text{Cough}) = 0.95_{0.1,0.03}$
$\mu_{P_2^2}(P_2, \text{Fatigue}) = 0.3_{0.1,0.15}$.

Symptom-Disease Knowledge Base (Q^2): This relation encodes expert knowledge linking symptoms to diseases.
$\mu_{Q^2}(\text{Fever}, \text{Common Cold}) = 0.4_{0.3,0.3}$, $\mu_{Q^2}(\text{Fever}, \text{Influenza}) = 0.95_{0.05,0.02}$, $\mu_{Q^2}(\text{Fever}, \text{Pneumonia}) = 0.85_{0.15,0.09}$
$\mu_{Q^2}(\text{Cough}, \text{Common Cold}) = 0.9_{0.1,0.05}$, $\mu_{Q^2}(\text{Cough}, \text{Influenza}) = 0.75_{0.15,0.1}$
$\mu_{Q^2}(\text{Cough}, \text{Pneumonia}) = 0.95_{0.1,0.04}$
$\mu_{Q^2}(\text{Fatigue}, \text{Common Cold}) = 0.65_{0.15,0.15}$, $\mu_{Q^2}(\text{Fatigue}, \text{Influenza}) = 0.88_{0.13,0.06}$, $\mu_{Q^2}(\text{Fatigue}, \text{Pneumonia}) = 0.98_{0.08,0.02}$

We compute the Patient-Disease relation $R^2 = P^2 \circ Q^2$ using three different compositions:

1. **Sup−i composition with Minimum t −norm:**
$R^2 = P^2 \circ_{min} Q^2$, where $\circ_{min}$ is the standard max-min composition.
2. **Sup−i composition with Product t −norm:**
$R^2 = P^2 \circ_i Q^2$, where the t −norm i is $i(a,b) = a \times b$.
3. **Inf−w_i composition with Probabilistic Sum s −conorm:**
$R^2 = P^2 \circ^u Q^2$, where the s-conorm u is $u(a,b) = a + b - a \times b$.

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4.2 Results and Uncertainty Propagation Analysis

The full results, including the core values r and the uncertainty spreads (ε and δ), are presented below. This allows for a complete analysis of both the diagnosis strength and the confidence in that diagnosis.

Table 1Patient-Disease Relation R^2

Patient	Disease	Sup$-i$ composition with Minimum t $-$norm	Sup$-i$ composition with Product t $-$norm	$Inf-w_i$ composition with Probabilistic Sum s $-$conorm
P_1	Common Cold	$0.65_{0.15,0.15}$	$0.585_{0.185,0.175}$	$0.91_{0.704,0.09}$
	Influenza	$0.88_{0.13,0.06}$	$0.8075_{0.1325,0.0863}$	$0.875_{0.4775,0.125}$
	Pneumonia	$0.90_{0.1,0.05}$	$0.882_{0.162,0.068}$	$0.975_{0.4185,0.025}$
P_2	Common Cold	$0.90_{0.10,0.05}$	$0.855_{0.175,0.076}$	$0.755_{0.68,0.245}$
	Influenza	$0.75_{0.15,0.10}$	$0.7125_{0.2025,0.1205}$	$0.916_{0.389,0.084}$
	Pneumonia	$0.95_{0.1,0.03}$	$0.9025_{0.18,0.0677}$	$0.94_{0.445,0.06}$

4.3 Interpretation and Comparative Analysis

Diagnostic Certainty:

$Sup-i$ composition with minimum t $-$norm**:** Provides a clear, "best-path" diagnosis. For patient P_1, identifies Pneumonia and Influenza as most likely (cores of 0.90 and 0.88). The uncertainty bounds are directly inherited from the strongest path, leading to moderate uncertainty [5]. $Sup-i$ composition with Product t $-$norm**:** This "pessimistic" composition produces **lower core values and wider uncertainty spreads.** For example,P_1's likelihood of Pneumonia drops from 0.90 to 0.882, and the support widens from (0.80, 0.95) to (0.72, 0.95). This reflects the multiplicative accumulation of uncertainty, making it a conservative choice for risk-averse decision-making. $Inf-w_i$ composition with Probabilistic Sum s $-$conorm**:** This "optimistic" composition yields **higher core values but at the cost of significantly amplified uncertainty as seen in the wide bounds for** P_1's common cold diagnosis $0.91_{0.704,0.09}$ .Despite this amplification of uncertainty, it **provides strong diagnostic discrimination between the most likely options, as demonstrated by the 0.10 core separation between Pneumonia and Influenza for** P_1**. This characteristic makes it ideal for sensitive screening applications where identifying all potential cases is prioritized, even with reduced precision.**While these qualitative observations provide initial insights, a rigorous quantitative analysis is essential to objectively compare composition performance. The section 4.4 introduces specialized metrics to quantify uncertainty propagation, diagnostic discrimination, and decision confidence.

4.4 Quantitative Metrics and Statistical Analysis

To provide an objective comparison between compositions beyond qualitative assessment, we introduce quantitative metrics measuring uncertainty propagation and diagnostic characteristics.

4.4.1 Quantitative Uncertainty Metrics [6]

Diagnostic performance is assessed with the help of core value, uncertainty range, and confidence index.

Table 2 Quantitative Comparison Metrics for Patient P_1's Diagnoses

Composition Type	Disease	Core Value	Uncertainty Range	Confidence Index*
$Sup-i$ (Min)	Common Cold	0.65	0.30	0.50
	Influenza	0.88	0.19	0.74
	Pneumonia	0.90	0.15	0.78
$Sup-i$ (Product)	Common Cold	0.59	0.37	0.43
	Influenza	0.81	0.21	0.67
	Pneumonia	0.88	0.23	0.72
$Inf-w_i$ (ProbSum)	Common Cold	0.91	0.79	0.51
	Influenza	0.88	0.61	0.55
	Pneumonia	0.98	0.45	0.68

*Confidence Index = Core Value / (1 + Uncertainty Range)

4.4.2 Analysis of Diagnostic Discrimination

To evaluate the compositions' ability to discriminate between diagnoses, we calculate separation metrics:

Diagnostic Separation Analysis for Patient P_1:

***Sup*$-i$ composition with minimum t $-$norm:**

Influenza-Common Cold separation: 0.23 (strong discrimination)
Pneumonia-Influenza separation: 0.02 (weak discrimination)
Pneumonia-Common Cold separation: 0.25 (strong discrimination)
Coefficient of variation: 14.01%

***Sup-i* with Product t $-$norm:**

Influenza-Common Cold separation: 0.22 (strong discrimination)
Pneumonia-Influenza separation: 0.07 (moderate discrimination)
Pneumonia-Common Cold separation: 0.29 (strong discrimination)
Coefficient of variation: 16.26%

***Inf-w$_i$* with Probabilistic Sum s $-$conorm:**

Influenza-Common Cold separation: 0.03 (weak discrimination)
Pneumonia-Influenza separation: 0.10 (strong discrimination)
Pneumonia-Common Cold separation: 0.07 (moderate discrimination)
Coefficient of variation: 4.54%

From the resultant output uncertainties calculated in Table 1, it is evident that the $Sup-i$ (Min) composition decreases uncertainty, the $Sup-i$ (Product) composition can either reduce or increase uncertainty, while the $Inf-w_i$ (ProbSum) composition consistently amplifies uncertainty. This aligns with the composition's theoretical features.

4.5 Discussion on Computational Complexity

The introduction of generalized $Sup-i$ and $Inf-w_i$ compositions within the $\varepsilon-\delta$ framework incurs a predictable computational overhead compared to the standard Max-Min composition. **Max-Min:** For relations of size $|X|$ times $|Y|$ and $|Y|$ times $|Z|$, the complexity is $O(|X||Y||Z|)$, with each operation being a simple min and max on real numbers. ***Sup*$-i$ and *Inf*$-w_i$:** The complexity remains$(|X||Y||Z|)$. However, each pairwise operation is more expensive. Instead of a single min operation, it requires 4 multiplications and 3 additions for the $Sup-i$ composition with the product t $-$norm (and similarly for the $Inf-w_i$ composition with s $-$conorm). The subsequent approximate supremum/infimum operations on $\varepsilon-\delta$ numbers are comparable in cost to the standard max/min.

This overhead is linear and manageable, as the number of basic arithmetic operations per element is still constant. The $\varepsilon - \delta$ formalism thus provides a computationally feasible pathway for implementing these more expressive, higher-order fuzzy relational compositions, as demonstrated by the practical computation of all composition types in the medical diagnosis example of this section without resorting to the prohibitive costs of general Type-2 fuzzy set operations.

4.6 Clinical Implications and Practical Considerations

4.6.1 Concrete Clinical Impact: These compositions are effective in different medical situations:

For Emergency and Critical Care Situations, $Sup{-}i$ with Product t −norm results is preferred as it results in moderate uncertainty changes. It also reduces core values. So it is suitable in emergency and critical situations where a complete diagnosis is necessary.

For Screening and Primary Care, use $Inf{-}w_i$ with Probabilistic Sum**.** The composition results in high core values. It clearly separates different diseases. These features make it perfect for a large population to identify more sick people. The high sensitivity minimizes the missed cases

For Chronic Disease Management, use Max-Min Composition. It gives low uncertainty. This is suitable for long-term diseases, where continuous tracking is required.

4.6.2 Enhanced Clinical Decision Framework

Analysis (Tables 1 and 2) reveals three practical decision-making approaches.

1. Max-min composition is best suited for high-certainty approaches in routine diagnostics. It provides stable, reliable results with a maximum confidence index and minimal growth in uncertainty [5]. For the patient P_1, a clear diagnosis of pneumonia (0.90) with low uncertainty and a maximum confidence index (0.78) is detected, with minimal growth in uncertainty.
2. $Sup{-}i$ Product composition is more suitable in complex cases where careful diagnostics are required. Core values are reduced and the uncertainty range increases, which results in low confidence level. For the patient P_2, $Sup{-}i$ product composition calculated a lower Pneumonia score than the other two methods. It naturally produces more cautious estimates.
3. $Inf{-}w_i$ ProbSum composition is useful for sensitive situations, such as early detection and large-scale population diagnosis, by identifying the maximum number of cases with clear differences between diseases. (0.10 separation as compared to 0.02 in max-min composition). It produces the highest core values, but the uncertainty range is also very high, resulting in a lower confidence index.

4.6.3 Key Limitations and Validation

Our approach has some limitations.

1. $Inf{-}w_i$ composition yields a wide range of uncertainty. Careful clinical interpretation of fuzzy results is required.
2. The final decision rules must be reset. Each composition method needs its own diagnosis threshold, as each composition has different core values.

5. Conclusion and Future Work: This paper establishes a comprehensive theoretical and computational framework for Second Order Fuzzy Relational Equations (SOFREs) using $Sup{-}i$ and $Inf{-}w_i$ compositions within the efficient $\varepsilon - \delta$ formalism. Our contributions include:

1. Definition of $Sup{-}i$ and $Inf{-}w_i$ compositions for second order fuzzy relations;
2. Complete computational procedures for t −norms and s −conorms within $[0,1]$;
3. Analysis of necessary algebraic properties with solvability conditions; and
4. A quantitatively verified numerical demonstration in medical diagnosis situation

The results reveal that these compositions offer semantically distinct, tunable modes of reasoning. The $Sup{-}\,i$ composition provides a conservative chaining of reasoning, which is suitable for applications that require risk-averse decision-making. The $Inf{-}w_i$ composition offers an optimistic alternative that provides strong diagnostic discrimination but amplifies uncertainty, making it ideal for sensitive screening applications where false negatives are a primary concern. The study extends the modeling flexibility of second order fuzzy calculus.

Future Research Directions

This article serves as a foundation for several research opportunities. This article will help in developing a complete solution for SOFREs and reduce uncertainty in multistage $Inf{-}w_i$ compositions. Additionally, discovering various t −norms and s −conorms within $\varepsilon - \delta$ framework could produce compositions with specialized profiles to establish practical application guidelines.

The methodology presented in this paper serves as the basis for advanced systems that address multilayered uncertainty within a granular computing framework [10]. This opens new opportunities for fuzzy relational calculus in complex, real-world applications with quantitatively predictable behavior.

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