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An Approach to Strengthen Some Particular Numerical Iterative Methods

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Abstract: In this paper, our main endeavor is to strengthen the usual routine root-finding strategy of a numerical iterative method (such as the Bisection method, the Regula-Falsi method, the Newton-Raphson method, etc.) of a given equation $f(x) = 0$. To achieve this goal, we first, frame this usual routine root-finding strategy as a theorem and referred to the *weak common root-finding strategy of a numerical iterative method (WCRFSNIM* - defined on the real axis*)*. We point out certain limitations and weaknesses of the *WCRFSNIM* theorem such as, a numerical iterative method lies in this theorem unable to provide: (i) an even number of real roots (in the open interval of the intermediate value theorem (IMVT - defined on the real axis)) of a given equations $f(x) = 0$; (ii) an even number of complex roots (in the open disk of the intermediate value theorem (IMVT - defined on the complex plane)) of a given equation $f(z) = 0$ and (iii) finalizing a proper choice of an initial approximation (PCIA) from a neighborhood point of such an odd or even number of real and complex roots. Further, we develop a comprehensive plan to extend the *WCRFSNIM* theorem in to the strong theorems, so as the above limitations and weaknesses ((i) – (iii)) of the *WCRFSNIM* theorem are overcome by these our developed extended strong theorems. These extended theorems we call *the strengthened common root-finding strategy of a numerical iterative method (SCRFSNIM* - defined on the Real Axis and next on the Complex Plane). To demonstrate the performance and effectiveness of the *SCRFSNIM* theorem (defined on the Real Axis) and next the *SCRFSNIM* theorem (defined on the Complex Plane), certain typical application-oriented examples are illustrated.
Mathematics Subject Classification: 65B99, 65G40.
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1. Introduction

The main aim of the field of numerical analysis is to design and analyze techniques for finding an approximate but accurate solution to the linear and nonlinear equations. In recent times, many researchers have focused on solving such equations using the various techniques of numerical methods, leading to the development of many mathematical models in Engineering, Mathematics, and other disciplines. The strategy of a numerical iterative method is a mathematical procedure that uses an initial value (pivot value) to generate a sequence of improving approximate solutions for a class of problems, in which the n^{th} approximation is derived from the $(n-1)^{th}$ approximation [1, 4, 11]. Inherently, historical evidence regarding some particular numerical methods reveals that the origin of the Secant method all the way back to the rule of Double False position described in the 18^{th} century B.C. In an ancient Egyptian mathematical text, Egyptian Rhind Papyrus, named after Alexander Henry Rhind, showed that the rule of Double False Position coincides with the secant method. The Secant method predated Newton's method and was most commonly referred to as the rules of Double False Position and it can be viewed as the first iteration of the Secant method. Newton's method was devised by Isaac Newton in 1669 and has been a cornerstone importance in numerical analysis for centuries. In 1690, English Mathematician Joseph-Raphson greatly simplified Newton's method to its current form. Every book on numerical methods has details of these methods, and recently, several research papers are making differing claims on their performance [6, 7, 11].

In this regard, note that there are severe limitations on analytical (direct) methods to find roots of equations of the form

$$f(x) = 0 \qquad (1)$$

where $f(x)$ is a polynomial of higher degree or an expression in the form either algebraic, transcendental or combination of both. To tackle such limitations, researchers have evolved graphical and numerical methods instead of the analytical or direct methods to solve equations of the form (1). It has been seen that the graphical methods, though simple, gives results to a low degree of accuracy. Hence graphical methods are useful for finding tentative nature of roots, as the nature of roots lie somewhere within the open interval on which the graph of the given equation is continuous. Consequently, graphical and numerical methods have increasing demand for numerical answers to various problems. In an age where time is money, the computational efficiency offered by these methods is a big plus. Thus, numerical methods have become indispensable equipment in hands of researchers of Mathematical Sciences. Furthermore, we notice that the usual routine process of finding roots of equation of the form (1) using a numerical iterative method (such as the Bisection method, the Regula – Falsi method, the Newton – Raphson method, etc.) is executed as follows: (i) first utilizing *the Intermediate Value Theorem (IMVT)* to the equation of the form (1) and then (ii) using a *proper choice of an initial approximation (PCIA)* in a neighborhood of the single root, that lies in the open interval of the IMVT theorem. Thus, we claim that a numerical iterative method works to find a proper approximate root of a given equation, utilizing the two indispensable components: the *IMVT* theorem and a *PCIA*. Moreover, our next and the most important claim is that there are some severe limitations and weaknesses in finding roots of equation of the form (1) using such a numerical iterative method. To locate and resolve such limitations and weaknesses, first of all, we frame the common (routine) root - finding strategy of a numerical iterative method as the theorem WCRFSNIM (on the Real Axis) as follows: **The theorem WCRFSNIM (on the Real Axis)** A numerical iterative method (such as the Bisection method, the Regula – Falsi method, the Newton – Raphson method, etc., which is compatible to come under this titled theorem), works to find an approximate but accurate root of a given equation $f(x) = 0$ if and only if first, its first component, IMVT, is applied to the given equation $f(x) = 0$, and then its next component, PCIA, is chosen properly (from the neighborhood of the single root that lies in the open interval of the IMVT).

In fact, from the present research point of view, our further aim is to address the following queries regarding the theorem WCRFSNIM (on the Real Axis): How does a numerical iterative method work to find a desired approximate root of a given equation? To execute on a complete scale, what are its strengths and limitations? How can the strengths be increased, and the limitations overcome? In order to tackle all these queries, we start our investigation as follows:

By a minute study of the details working of a numerical iterative method (lies in the theorem WCRFSNIM (on the Real Axis)) to find a proper root of a given equation, we identify its limitations and weaknesses as follows: first of all, its component the IMVT executes, but it fails to provide an even number of real and also complex roots. Furthermore, it is difficult to select a PCIA in the neighborhood of "an odd number of real roots and an even number of real or complex roots". Due to such failures, we recognize the theorem WCRFSNIM as a weak theorem. To overcome such weaknesses and limitations of the theorem WCRFSNIM (on the Real Axis) it necessitates to modify or improve this theorem. Hence, we are motivated to write this research paper as an affirmative response to these challenges. Thus, our further research aims to extend the weak theorem WCRFSNIM (on the Real Axis) so that it transforms into the stronger or strengthen theorem SCRFSNIM (on the Real Axis and also on the Complex Plane). This stronger or strengthen theorem overcomes weaknesses and limitations of the weak theorem.

We have organized further work of this paper in the following sequence: Section 2: Necessary prerequisites are included. Section 3: Objective of the research work is prescribed. Section 4: the stepwise working of the theorem SCRFSNIM (on the Real Axis and also on the Complex Plane) with typical application-oriented examples are demonstrated. Section 5: the conclusion is presented, followed by the references.

2. Prerequisites

i) Nature of Roots depending on the Nature of the Discriminant of a quadratic Equation: The quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$, where a, b, c are real numbers, has real roots if the discriminant D is non-negative i.e. D ≥ 0. Also note that if $D \neq 0$, then the roots are distinct. On the other hand, if $D = 0$, then the roots are equal. Furthermore, if the discriminant D is negative, i.e., D < 0, then it has two complex roots.

ii) Discartes' Rule of signs: The equation $f(x) = 0$ cannot have more positive roots than the changes of signs in $f(x)$, nor more negative roots than the number of changes of signs in $f(-x)$. For instance, see the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$. Here we see that the number of sign changes of $f(x)$ is 3, which shows that it has no more than 3 positive roots. Also, we see that the number of sign changes of $f(-x) = -2x^7 + x^5 - 4x^3 - 5 = 0$ is 2, which shows that it has not more than 2 negative roots.

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iii) Existence of Imaginary Roots: If an equation of nth degree has at most p positive roots and the most q negative roots, then it follows that the equation has at least $n-(p+q)$ imaginary roots. For instance, consider the equation $f(x) = 2x^7 - x^5 + 4x^3 - 5 = 0$. Here we see that, sign changes of $f(x)$ are 3, which shows that it has not more than 3 positive roots. Also, we see that the number of sign changes in $f(-x) = -2x^7 + x^5 - 4x^3 - 5 = 0$ is 2, which shows that it has no more than 2 negative roots. Evidently, the given equation is of the 7th degree, having at most 3 positive roots and 2 negative roots. Hence, existence of imaginary roots signifies the given equation has 2 imaginary roots.

iv) Physical Appearance of a Given Equation /Table Values of $f(x) = 0$ for various values of x: By taking into account particular values of x at and around the origin, and then observing the corresponding values of $f(x)$, if $f(x_i)\,f(x_j) < 0$, for any nearest i and j, then there exists a root $\xi\epsilon(\,x_i,\,x_j)$. For illustration, we guess the roots of a given equation by observing its physical appearance. For instance, consider the equation $f(x) = x^2 - 1 = 0$, has roots ∓1. or $f(0)$ is negative and $f(2)$ is positive, and $f(0)\,f(2) < 0$, hence the root must lie in $(0, 2)$.

v)Bifurcating or Splitting a Given Equation: Consider an equation $f(x) = 0$. We bifurcate this equation as $f_1(x) = f_2(x)$; then, the number of intersection points of graphs of these two equations $y_1 = f_1(x)$ and $y_2 = f_2(x)$ give the same number of roots of the given equation.

vi) Complex Plane or Z plane: We can represent a complex number $z = x + iy$ by a point in the XY plane. This plane is called "the complex plane or Z plane".

vii) A Connected Set: An open set S is said to be connected if any two points of the set can be joined by a path consisting of straight-line segments (i.e. a polygonal path), all points of which lie in S. (See Klir)

viii) A Domain or an open region: An open, connected subset of the set of complex numbers ℂ (or the complex plane) is often called a domain or region.

ix) An open Disk: For any $z_0 = x_0 + iy_0$, an open ball or open disk with center z_0 and radius r, is denoted by the symbol B (z_0, r) and expressed as

B $(z_0, r) = \{\, z = x + iy \colon |z - z_0| < r\}$, represents the set of all points z in the complex plane that lie within the disk.

3. Objective of the Study

In this study, our main objective is to overcome the limitations and weaknesses of a numerical iterative method that lies in the theorem WCRFSNIM, for finding an approximate but accurate root of a given equation. The theorem WCRFSNIM reveals that the root-finding process of its any numerical iterative method is carried out through the collaboration of components: the IMVT and a PCIA. Furthermore, we claim that these components of a numerical iterative method have certain weaknesses and limitations for finding a proper approximate root of a given equation. Hence, we investigate the weaknesses and limitations of these components and suggest remedies to eliminate them.

3.1. Investigation of limitations and weaknesses of the theorem IMVT and strategies to overcome them.

We first give existing statement the theorem IMVT and then make investigation to address its limitations and weaknesses as follows:

3.1(a). Statement of theorem IMVT [7] If a function $f(x)$ is continuous on some interval [a, b] and $f(a)f(b) < 0$, then the equation $f(x) = 0$ has at least one root or an odd number of roots in (a, b).

We point out the following important limitations and weaknesses of this theorem:

(i) It does not provide a single root or an even number of real roots in its open interval, since the condition $f(a)f(b) > 0$ has not been included in the theorem. For illustration see equations (2) and (3) and their respective Figure (1) and (2).

(ii) It unable to provide complex roots of the given equation, since theorem IMVT is stated on the Real Axis. Also, the condition $f(a)f(b) > 0$ is excluded from the theorem. For illustration see equations (4) and (5) and their respective Figure (3) and (4).

To overcome the weakness (i), we include the extra condition $f(a)f(b) > 0$ in the IMVT, and we call this new theorem the extended Intermediate Value Theorem (EIMVT) defined on the Real Axis. We propose this theorem as follows:

3.1(b). The theorem EIMVT (on the Real Axis)

If a function $f(x)$ is continuous on some interval [a, b] and either $f(a)f(b) < 0$ or $f(a)f(b) > 0$, then the equation $f(x) = 0$ has either a single root, an odd number of roots, or an even number of roots in (a, b).

Note: This new theorem, the theorem EIMVT (on the Real Axis), is an extension of the original theorem, the theorem IMVT. It acts as the Extended and Generalized Intermediate Value Theorem, which overcomes the weakness (i).

Next, we proceed to overcome the weakness (ii), i.e. the theorem IMVT and also the theorem EIMVT, defined on the Real Axis, are unable to provide complex roots of a given equation. We address this failure by formulating the theorem EIMVT (on the Complex Plane) as follows:

3.1(c). The theorem EIMVT (on the Complex Plane)

If $f(z)$ is a complex- valued function of a complex variable z, continuous on some domain $|z - z_0| \le r$ of the complex plane and it intersects the Real Axis, the Imaginary Axis or itself in the domain $|z - z_0| < r$, then the equation $f(z) = 0$ has purely real, purely imaginary or complex roots, respectively, in the domain $|z - z_0| < r$ of the complex plane.

We call to this new theorem the extended Intermediate Value Theorem (EIMVT) defined on the complex plane. This new theorem the EIMVT (on the complex plane) acts as the Extended and Generalized Intermediate Value Theorem, which overcomes the weakness (ii).

3.2. Investigation of limitations and weakness of choosing a PCIA component and strategy to overcome it

In this context, we notice that it is easy to make a PCIA in the neighborhood of a single root that lies within the open interval of the IMVT (on the Real Axis), the open interval of the EIMVT (on the Real axis) and the open disk of EIMVT (on the Complex plane); but it is complicated to make a PCIA in the neighborhood of an odd or even number of roots that lie within the open interval of the IMVT (on the Real Axis), the open interval of the EIMVT (on the Real Axis) and open disk of the EIMVT (on the Complex Plane). We overcome these limitations and weakness, by including a criterion to locate a Single Root (CLSR) (such as physical appearance, table values of $f(x) = 0$ for various values of x, nature of discriminant of a quadratic equation, existence of real and imaginary roots, bifurcating or splitting a given equation, etc.) as the third working component of the required numerical method lies in the theorem WCRFSNIM (on the Real Axis).

At this stage, we are in a position to overcome the limitations and weaknesses of components the IMVT and a PCIA by combining the above section 3.1 and section 3.2 concisely. The process of combining these two sections is nothing but transferring the theorem WCRFSNIM (on the Real Axis) into the extended and generalized SCRFSNIM Theorem (on the Real Axis) and the SCRFSNIM Theorem (on the Complex Plane). As a result, we find that a numerical iterative method lies in the SCRFSNIM Theorem (on the Real Axis and also on the Complex Plane) acts as an extended numerical iterative method having extended working components, with ability to provide all types of alternative roots. We invent these two theorems as follows:

3.3(a). The extended and general theorem SCRFSNIM (on the Real Axis)

An extended numerical iterative method (such as the extended Bisection method, the extended Regula- Falsi method, the extended Newton-Raphson method, etc., which is compatible with this titled theorem) works to find an approximate but accurate root of a given equation $f(x) = 0$ if and only if first, its first component, the EIMVT (on the Real Axis) is applied to the given equation $f(x) = 0$ and then its third component, the CLSR (such as physical appearance, table values of $f(x) = 0$ for various values of x, nature of the discriminant of a quadratic equation, etc.) is applied to the given equation $f(x) = 0$, so that the second component, a PCIA is easily made as the usual practice.Note *that the* application of the CLSR to a given equation $f(x) = 0$, *locates a single real root from the open subinterval of the open interval of the EIMVT (on the Real Axis) having an odd or even number of real roots. Hence, choosing a PCIA in a neighborhood of the single root is trivial and the required extended numerical iterative method starts to work with this PCIA for finding the proper approximate but accurate real root of the given equation.* Note that this newly invented extended and generalized theorem SCRFSNIM (on the Real Axis) is an extension of the theorem WCRFSNIM (on the Real Axis). Our claim is, this new Theorem SCRFSNIM (on the Real Axis) has potential to overcome the limitations/weaknesses of the theorem WCRFSNIM (on the Real Axis).Furthermore, we point out one more weakness of the general and extended theorem SCRFSNIM (on the Real Axis), it is unable to provide complex roots of the given equation. To overcome this limitation, we have invented the general and extended theorem, SCRFSNIM (on the Complex Plane) as follows:

3.3 (b) The general and extended theorem SCRFSNIM (on the Complex Plane)

An extended numerical iterative method (such as the extended Secant and the extended Newton-Raphson method which are compatible with this titled theorem) works to find an approximate yet accurate complex root of a given equation $f(z) = 0$ if and only if first, its first component, the EIMVT (on the Complex Plane) is applied to the given equation $f(z) = 0$ and then its third component, the CLSR (such as: Physical Appearance, Nature of Discriminant of a Quadratic Equation,

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Existence of Real and Imaginary Roots, Bifurcating or Splitting a Given Equation, etc.) is applied to the equation $f(z)=0$, so that the second component a proper choice of PCIA is easily made as the usual practice.

Note *that the* application of the CLSR to the given equation $f(z)=0$, *locates a single complex root in the "open subinterval or open sub-disk" of the "open interval or open disk" of the EIMVT (on the Complex Plane) having an even number of complex roots). Hence, choosing a PCIA in the neighborhood of the single root is trivial and the required extended numerical iterative method starts to work with this PCIA to find the proper approximate yet accurate complex root of the given equation.* Note that this newly invented extended and generalized theorem SCRFSNIM (on the Complex Plane) is an extension of the theorem SCRFSNIM (on the Real Axis). Our claim is, this new theorem SCRFSNIM (on the Complex Plane) has potential to overcome all the limitations or weaknesses of the theorem SCRFSNIM (on the Real Axis) and also the theorem WCRFSNIM (on the Real Axis).

4. Stepwise Working of Theorems: SCRFSNIM (on the Real Axis) and SCRFSNIM (on the Complex Plane)

A required extended numerical iterative method that lies in the Theorem SCRFSNIM (on the Real Axis) and next in the Theorem SCRFSNIM (on the Complex Plane) finds a proper approximate real and complex root, respectively, of a given equation through the collaboration of their working components. A detailed working procedure of these theorems is prescribed in the following steps:

Step 1. Strategy for finding particularly an even number of real roots of an equation $f(x)=0$ and an even number of complex roots of an equation $f(z)=0$

An extended numerical iterative method that lies in the extended and generalized theorem SCRFSNIM (on the Real Axis) works to find a root of a given equation $f(x)=0$ as follows: First, its first component, the EIMVT (on the Real Axis) is utilized to the given equation $f(x)=0$, yields all possible alternative real roots, i.e., either a single root, an odd number of roots, or an even number of roots within the open interval of the EIMVT (on the Real Axis). Then, the extended numerical iterative method has to find a proper approximate real root of the given equation $f(x)=0$ from these all-possible alternative real roots. Here, we consider the case of finding a desired approximate root from only *an even number of real roots* (because the case of an odd number of roots will be handled in exactly the same way as that of the even number of roots; and the case of single root is handled as a usual or routine practice). In this case, from the nature of the graph of the theorem EIMVT (on the Real Axis) for the given equation $f(x)=0$, we observe that the graph intersects the X- Axis, these intersection points are nothing but the even number roots of the equation, and lie within the open interval of the EIMVT (on the Real Axis). See: just like figures 1 and 2.

Proceeding similar to the above paragraph, an extended numerical iterative method that lies in the generalized and extended theorem SCRFSNIM (on the Complex Plane) works to find a root of a given equation $f(z)=0$ as follows: First, its first component, the EIMVT (on the Complex Plane) is applied to the given equation $f(z)=0$, yields an even number of complex roots within the open disk of the EIMVT (on the complex plane). In this case, from the nature of the graph of the theorem EIMVT (on the complex plane) for the given equation $f(z)=0$, we observe that the graph does not intersect the X−Axis, and hence an even number of roots lie either in the upper half or the lower half of the complex plane; that is, these roots are complex and lie within the open disk of the EIMVT (on the Complex Plane). Furthermore, these roots can be divided into two subcases:

- case (i): If the graph of the given equation intersects the $y-Axis$, then it has an even number of purely imaginary roots, which lie within the open interval of the EIMVT (on the imaginary axis). See: just like figure 3.
- case (ii): If the graph of the given equation does not intersect the $y-Axis$, then it lies somewhere in the complex plane in a bifurcated form, that is, the given equation $f(x)=0$ is bifurcated as $f_1(x)=f_2(x)$. We observe from the nature of the graphs of the bifurcated equations that the number of intersections of these bifurcated graphs indicate the even number of roots, which are complex and lie within the open disk of the EIMVT (on the Complex Plane). See: just like figure 4.

Step 2. Locating a single root from an even number of real and next from an even number of complex roots

It is noticed that making a PCIA in a neighborhood of an odd and even number of real and complex roots, respectively, is complicated using a numerical iterative method that lies in the theorem WCRFSNIM (on the Real Axis). We simplify this complicated task by using a generalized and extended numerical iterative method that lies in the generalized and extended theorem SCRFSNIM (on the Real Axis) and next in the generalized and extended theorem SCRFSNIM (on the Complex Plane). For, we proceed as follows: We see from the above **Step1**, i.e., in short, the component EIMVT (on the Real Axis) of a required extended numerical iterative method (that lies in the the generalized and extended theorem SCRFSNIM (on the Real Axis)) is applied to the given equation $f(x)=0$. This yields an even number of real roots in the open interval of the EIMVT (on the Real Axis). Now, we apply the component CLSR to the given equation $f(x)=0$ (having the even number of real roots in the open interval of the EIMVT (on the Real Axis)). As a result, we get location of a single root in the open subinterval, say (a', b') of the open interval, say (a, b) of the EIMVT (on the Real Axis).

Similarly, we see from the same **Step1**, i.e, in short, the component EIMVT (on the Comple Plane) of a required extended numerical iterative method (that lies the the generalized and extended theorem SCRFSNIM (on the Comple Plane)) is applied to the given equation $f(z)=0$. This yields an even number of complex roots in the open interval of the EIMVT (on the Complex Plane). Now, we apply second component the CLSR to the given equation $f(z)=0$ (having the even number of complex roots in the open interval of the EIMVT (on the Complex Plane)). As a result, we get location of a single root in the open sub-disk $|z|<r',(where\, r'<r)$ of the open disk $|z|<r$ of the EIMVT (on the Complex Plane).

Step 3. Finding a proper approximate root using a PCIA in the neighborhood of a single root

As the usual routine practice, it is very natural to say that:

(i) A PCIA can be easily chosen in a neighborhood of a single root that lies within the subinterval (a', b') and also within the open sub disk $|z|<r'<r$.

(ii) Utilizing this PCIA lies within the subinterval (a', b'), an appropriate extended numerical iterative method that lies in the extended and generalized theorem SCRFSNIM (on the Real Axis) works smoothly to find the desired approximate root of the given equation $f(x)=0$.

Similarly, using the PCIA that lies within the open sub disk $|z|<r'<r$), an appropriate extended numerical iterative method that lies in the extended and generalized theorem SCRFSNIM (on the Complex Plane) work smoothly to find the desired approximate root of the given equation $f(z)=0$.

From the above discussion, it is evident that there is no need to worry, whether a proper approximate real or complex root of a given equation is to find from the open interval having an odd or even number of real or complex roots.

Applications of the Theorems: SCRFSNIM (on the Real Axis) and SCRFSNIM (on the Complex Plane)

Let us examine applications of the Theorem SCRFSNIM (on the Real Axis) and the Theorem SCRFSNIM (on the Complex Plane) for finding an approximate yet accurate root of the following application-oriented equations (2) to (5), utilizing the stepwise procedure described in Section-4:

$$y = x^2 = 0 \qquad (2),$$
$$y = x^2 - 1 = 0 \qquad (3),$$
$$y = x^2 + 1 = 0 \qquad (4) \text{ and}$$
$$y = x^2 + 2x + 2 = 0 \qquad (5).$$

Examples (2) and (3) are solved using the extended and generalized theorem SCRFSNIM (on the Real Axis), and examples (4) and (5) are solved using the extended and generalized theorem SCRFSNIM (on the Complex Plane).

First, we solve examples (2) and (3):

Step 1. Strategy for finding the single root and even roots of the given equations (2) and (3), respectively

A required extended numerical iterative method that lies in the extended theorem SCRFSNIM (on the Real Axis) finds the single root of the given equation (2) as follows: First of all, its component EIMVT (on the Real Axis) is applied to the given equation (2), as a result it is continuous on the interval $[a=-1, b=1]$ and we get $f(a=-1).f(b=1)>0$. Next, its second component CLSR (physical appearance of the given equation and table values of $f(x)=0$ for various values of x) is applied to the given equation (2) i.e. $f(x)=x^2=0$, which yields a single root 0 (zero) within the $(a=-1,\ b=1)$. See Figure 1.

Proceeding similar to the above paragraph, a required extended numerical iterative method that lies in the extended theorem SCRFSNIM (on the Real Axis) finds the even root(s) of the given equation (3) as follows: First of all, its component EIMVT (on the Real Axis) is applied to the given equation (3), as a result it is continuous on the interval and $[a=2,\ b=-2]$, and we get $f(a=-2).f(b=2)>0$. Next, its second component CLSR (nature of discriminant of a quadratic equation and table values of $f(x)=0$ for various values of x) is applied to the equation (3) i.e. $f(x)=x^2-1=0$, which yields even roots ±1 within the $(a=2,\ b=-2)$. See Figure 2.

Step 2 and step 3. Finding an approximate but accurate root from the open interval having single and even root(s) of the given equation (2) and (3), respectively

First, for equation (2): From the above step 1, we see that equation (2) has a single root zero (0) lies in the open interval $(a,b)=(-1,1)$. Choosing a PCIA in the neighborhood of this single root $0\in(-1,1)$ is very simple, as the usual practice. The required extended numerical iterative method from the theorem

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SCRFSNIM (on the Real Axis) works smoothly with this PCIA to find an approximate and accurate root of the given equation. Performing the iteration process sufficiently large number of times, we obtain an approximate and accurate root of the given equation.

Next, for equation (3): From the above step1, we see that the equation (3) has two distinct real roots ± 1, which lie in the open interval $(a = 2, b = -2)$. Now, we apply second component, the CLSR (physical appearance or table values of $f(x) = 0$ for various values of x) to the given equation $x^2 - 1 = 0$. From table values of $f(x) = 0$ for various values of x, we find the single root, i.e., $-1 \in (-2, -0.5)$, since $f(-2)\, f(-0.5) < 0$. We select any neighborhood point of -1$\in (-2, -0.5)$, as the PCIA. We apply the required extended numerical iterative method from the theorem SCRFSNM (on the Real Axis) using this PCIA to find the desired root of the given equation. After performing the first, second, and subsequent iterations, we obtain an approximate but accurate root that is sufficiently close to -1.

Note that by proceeding in the same manner as above, we obtain another desired approximate root that is sufficiently close to 1.

Next, we solve examples (4) and (5):

Step 1. Strategy for Finding even roots of equation (4) and next equation (5)

An extended numerical iterative method that lies in the extended theorem SCRFSNIM (on the Complex Plane) finds the even roots of the equations (4) and (5) as follows: First, its component, EIMVT (on the Complex Plane) is applied to the equations (4) and then (5). As a result, equations (4) and (5) are continuous on some closed disk $|z - z_0| \le r$ of the complex plane and each has even number of roots in the in the open disk $|z - z_0| < r$.

In particular, the equation (4) is continuous over the closed interval $[a = -2i, b = 2i]$ on the imaginary axis. Next, its component, the CLSR (namely, nature of the discriminant of quadratic equation) is applied to the equation (4), i.e., $f(x) = x^2 + 1 = 0$, which yields roots $\pm i$ within the $(a = -2i,\ b = 2i)$. See Figure 3.

Similarly, the equation (5) is continuous on the closed disk $|z - (-1 \pm i)| \le 2$ of the Complex Plane. Next, its component, the CLSR (namely, nature of discriminant of quadratic equation) is applied to the equation (5) i.e. $f(x) = x^2 + 2x + 2 = 0$, which yields roots $x =$ -1$\pm i$ in the open disk $|z - (-1 \pm i)| < 2$. Furthermore, if we bifurcate equation (5) and sketch the graphs of the resulting equations, we observe that they intersect each other, this point of intersection is nothing but a root of the equation (5). Specifically, we bifurcate the equation (5) i.e. $f(x) = x^2 + 2x + 2 = 0$, into $f_1(x) = \frac{x^2}{-2}$ and $f_2(x) = x + 1$. The graph of these two equations is expected to intersect at the points -1$\pm i$. Thus, roots of the equation (5) are complex i. e. -1$\pm i$, which lie in the open disk $|z - (-1 \pm i)| < 2$. See Figure 4 (a) and 4 (b). Note that similar to Figure 4 (a), Figure 4 (b) can be drawn.

Note that: The roots of the equation $x^2 + 2x + 2 = 0$ or equivalently $f_1(x) = \frac{x^2}{-2} = x + 1 = f_2(x)$ are complex, that are, -1$\pm i$, which indicate no real intersection points. Moreover, in the Figure 4 (a), 1$\pm i \in |z - (-1 \pm i)| < 2$, which satisfies our need of selecting a PCIA in the neighbourhood of the single root 1$\pm i$ in this open disk.

Step 2 and step 3. Finding an approximate but accurate root from the open interval and open disk containing the even roots of equations (4) and (5), respectively

First, for equation (4): From the above step 1, we see that equation (4) has even (two) complex roots $x = \pm i$ that lie in the $(a = -2i,\ b = 2i)$. Now, applying the CLSR component (such as physical appearance and table values of $f(x) = x^2 + 1 = 0$ for various values of x) to the equation (4), i.e., $f(x) = x^2 + 1 = 0$, we find $f(0.5i) f(1.5i) < 0$. Hence, the single root $i \in (0.5i,\ 1.5i)$. We choose any neighborhood point of this single root i as a PCIA. Then the required extended numerical iterative method from the theorem SCRFSNIM (on the Complex Plane) with this PCIA is used to find the desired root of the given equation. After performing the first, second, and subsequent iterations (*i.e.*, a sufficient number of iterations), we obtain an approximate and accurate root that is, sufficiently close to i.

Similarly, by proceeding as above, we obtain the next desired approximate root that is sufficiently close to $-i$.

Next, for equation (5): From step1 above, the equation (5) has complex roots $x =$ -1$\pm i$. Now applying the component CLSR (such as nature of discriminant of a quadratic equation and bifurcation of the given equation) to the equation (5) i.e. $f(x) = x^2 + 2x + 2 = 0$, we find a single root -1$+i \in |z - (-1 + i)| < 2$. We choose any neighborhood point of this single root $-1 + i$ as a PCIA. Now, the extended numerical iterative method from the theorem SCRFSNIM (on the Complex Plane) with this PCIA is used to find a desired root of the given equation. After performing first, second, and subsequent (sufficient number of) iterations, we obtain an approximate and accurate root which is sufficiently close to $-1 + i$.

Similarly, proceeding as above, we obtain a desired approximate root that is sufficiently close to $-1 - i$, i.e., $-1 - i \in |z - (-1 - i)| < 2$.

Note that, to find the complex root(s) of a given equation, only the Secant and Newton--Raphson methods from the generalized and extended theorem - SCRFSNIM (on the Complex Plane) are applicable.

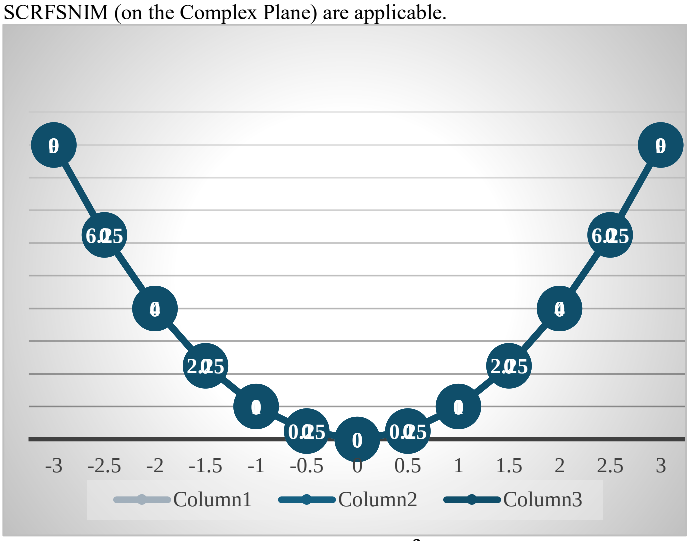

Figure 1: Graphical representation of $y = x^2$.

In the figure 1, a root $0 \in (1, -1)$. Hence, we choose an initial approximation in this open interval.

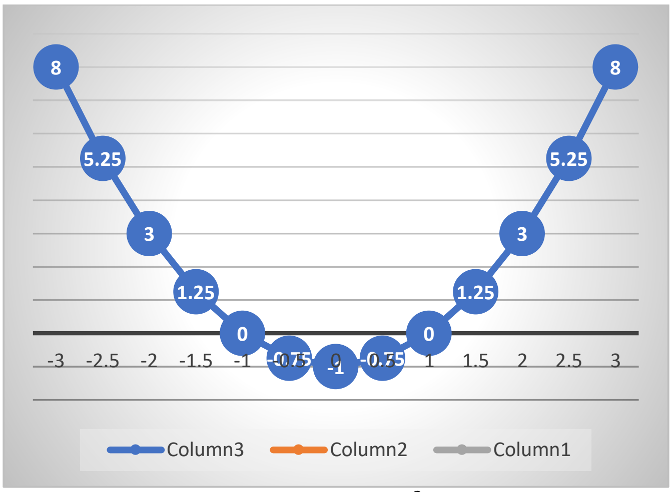

Figure 2: Graphical representation of $y = x^2 - 1$.

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In the figure 2, a root $1 \in (0,\ 2)$. Hence, we choose an initial approximation in this open interval.

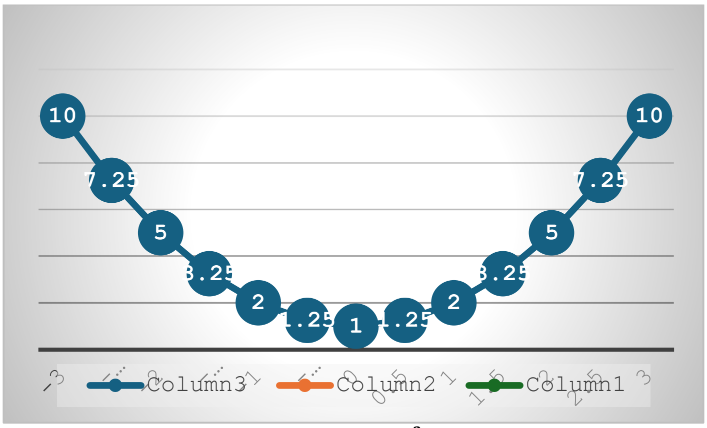

Figure 3: Graphical representation of $y = x^2 + 1$.
In the figure 3, a root $i \in (0,\ 2)$. Hence, we choose an initial approximation in this open interval.

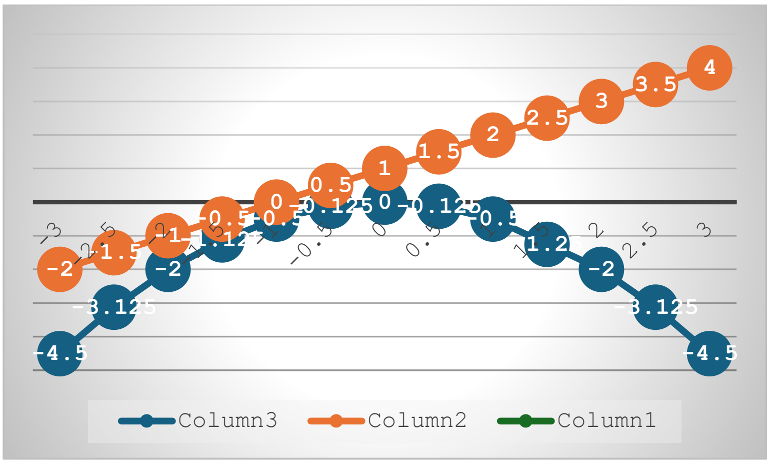

Figure 4 (a): The parabola formed by blue circles and straight-line formed by the orange circles represent the graph of $f_1(x) = \frac{x^2}{-2}$ and $f_2(x) = x + 1$ respectively.
In the figure 4(a), a root $-1-i \in |z-(-1-i)| < 2$. Hence, we choose an initial approximation in this open disk.
Proceeding similarly to Figure 4 (a), we can stress Figure 4 (b): graph of $f_1(x) = x^2$ and $f_2(x) = -2x - 2$, in which, the root $-1+i \in |z-(-1+i)| < 2$.
Thus, the above illustrated examples (2) to (5) reveal that the inabilities (such as providing an even number of real and complex roots, selecting a PCIA in the neighborhood of an odd or even number of real or complex roots, and hence finding an appropriate root of a given equation from an odd or even number of real or complex roots) of a numerical iterative method that belongs to the theorem WCRFSNIM (defined on the Real Axis) are overcome by utilizing a newly proposed extended numerical iterative method that belongs to the theorems: SCRFSNIM (defined on the Real Axis) and SCRFSNIM (defined on the Complex Plane).

5. Conclusion

In this paper, first of all, the usual routine root-finding strategy of a numerical iterative method (such as the Bisection method, the Regula-Falsi method, the Newton-Raphson method, etc.) that works to find a proper root of a given equation, was framed as the WCRFSNIM theorem (defined on the Real Axis). It has been claimed that a numerical iterative method that lies in the WCRFSNIM theorem have certain severe limitations and weaknesses due to its working components: first, the IMVT theorem (defined on the Real Axis), and second, a PCIA. To overcome the limitations and weaknesses of the first component IMVT theorem (defined on the Real Axis), we extended it in two ways: (i) to the EIMVT theorem (defined on the Real Axis) and (ii) to the EIMVT theorem (defined on Complex Plane). And to address the limitations and weaknesses of the second component a PCIA, we proposed a CLSR as a third working component.

Further, utilizing the above three newly invited working components appropriately, we proposed two new strong root finding strategies as the extended and generalized theorems: (i) the theorem SCRFSNIM (defined on the Real Axis). (ii) the theorem SCRFSNIM (defined on the Complex Plane). An extended and generalized numerical iterative method (such as the extended Bisection method, the extended Regula-Falsi method, the extended Newton-Raphson method, etc.) from both of these extended and generalized theorems overcome all the limitations and weaknesses identified in the theorem WCRFSNIM (defined on the Real Axis).

Further, numerical applications of the newly proposed theorems have been presented to show its expediency and effectiveness.

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