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Dynamics of Complexity of a Discrete – Time Prey-Predator system

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Abstract
Investigations were carried out on a discrete-time prey-predator system. The characteristic pattern within periodic windows appearing in the chaotic region of bifurcation of this proposed prey-predator system indicates that the system is of a complex nature. This implies that the system is internally composed of a multicomponent structure. Such components evolve independently, and to understand the evolution of these components, one has to follow the rule of probability. Topological entropy is a measure of complexity; "a higher topological entropy indicates a more complex system." During the process of study, fixed points are calculated and their stability criteria discussed in detail. Bifurcation diagrams of the system obtained by varying the parameter and, also, a typical region of the bifurcation displaying a periodic window magnified. A complex pattern of scenarios observed within a periodic window is discussed with proper justification. Numerical work is performed to get attractors of regular and chaotic motion, followed by calculations of Lyapunov exponents. Numerical calculations were again extended to find the topological entropies, *a measure of complexity*, of the system, which were displayed graphically. Finally, correlation dimensions for some chaotic attractors were also calculated. The results obtained through this investigation are interesting and very significant.

Key Words: Lyapunov Exponents, Topological Entropy, Chaos, Bifurcation.

1. Introduction

Some of the nonlinear systems are internally composed of multicomponents. During evolution, these components do not follow a single rule, and each component evolves independently in its own way. Such nonlinear systems are called complex systems. The evolution of such a multicomponent of a complex system is measured by using the rule of probability called complexity measure. It is actually the increase in topological entropy of the system. The system is considered to be more complicated when there is a rise in the topological entropy of the system. Adler, R. L., Konheim, A. G., & McAndrew, M. H.[2] Baldwin, S. L., & Slaminka, E. E.[4], Balmforth, N. J., Spiegel, E. A., & Tresser, C.[5], Beddington, J. R., Free, C. A., & Lawton, J. H.[7], Bowen, R.[9], Elsadany, A. E. A.[12], Gribbin, J.[16], Gribble, S.[17], Gribble, S.[18], Heffernan, D. M.[22], Iwai, K.[23], Nagashima, H.[27], Saha, L. M. Saureesh Das, Til Prasad Sarma, Anunay Choudhary, Das, M. K.[30], Smith, J. M.[36], Walby, S [35], Weaver, W. [36]. The rate of mixing in a dynamical system is measured by the topological entropy. In fact, it calculates the exponential rate at which the number of distinct orbits increases over time. Topological entropy is also called Kolmogorov-Sinai entropy, Gribble, S.[17]. A complex system, like many other nonlinear systems, evolves into chaos in some parameter space but also produces a significant increase in topological entropy. The presence of complexity can be viewed through bifurcation patterns. Chaos in the system is measured by Lyapunov exponents (LCEs); a system evolves regularly or chaotically depending, respectively, on whether LCE > 0 or LCE < 0, Walby, S [35], Heffernan, D. M.[22], Beddington, J. R., Free, C. A., & Lawton, J. H.[7], Bowen, R.[9], Benettin, G., Galgani, L., Giorgilli, A., & Strelcyn, J. M.[8], Bryant, P., Brown, R., & Abarbanel, H. D.[10], Grassberger, P., & Procaccia, I [14], Grassberger, P., & Procaccia, I [15], Hadeler, K. P., & Freedman, H. I[20], Iwai, K. [23].

In recent times, numerous prey-predator problems of nonlinear dynamics have appeared in the literature. Each of these problems is considered in relation to different environmental situations and conditions. Some of the problems are also influenced by other agents. The results obtained by these studies are very important and provide significant direction for further studies, Saha, L. M. Saureesh Das, Til Prasad Sarma, Anunay Choudhary, Das, M. K.[30], Abrams, P. A., & Ginzburg, L. R.[1], Bascompte, J., Solé, R. V., & Valls, J.[6], DeCoster, G. P., & Mitchell, D. W. [11], Grafton, R. Q., & Silva-Echenique, J.[13], Gümüş, Ö. A. [19], Harikrishnan, K. P. [21], Liu, X., & Xiao, D. [24], Neubert, M. G., & Kot, M. [29], Saha, L. M., Dixit, P., & Erjaee, G. H. [31], Smith, J. M. [33], Tang, S., & Chen, L. [34], Xiao, Y., Cheng, D., & Tang, S. [37].

The current study's goal is to determine the complexity measures of the Dynamics's evolution of prey-predator's discrete-time system used to describe planktonic dynamics, Neubert, M. G., & Kot, M. [28]. Describing the mathematical formulation of the model, fixed points and their stability criteria are discussed in detail. As the calculation for stability corresponding to one of the fixed points appeared complicated, for simplicity, both parameters were equated for further calculations. The bifurcation diagram of the model displays periodic windows within which a complex pattern of scenarios is observed. This is due to the system's inherent complexity. Both chaotic (weird) and regular attractors are obtained and presented through graphics. Corresponding Lyapunov exponents (LCEs) were calculated. Numerical simulations were further advanced to obtain the topological entropy and correlation dimensions of chaotic attractors. Graphical representations of the results provide significant information about the evolutionary behavior of the system.

2. Model Description:

The Model:

$$x_{n+1} = \mu\, x_n (1 - x_n - y_n),$$
$$y_{n+1} = \beta\, x_n\, y_n \qquad (1)$$

Jacobian matrix of this system is

$$J = \begin{pmatrix} \mu(1-2x-y) & -\mu x \\ \beta y & \beta x \end{pmatrix} \qquad (2)$$

Fixed points of equation (1) are obtained as

$$P_1^*(0,\ 0)\ ,\ P_2^*\left(\frac{\mu-1}{\mu},\ 0\right),\ P_3^*\left(\frac{1}{\beta}\ ,\ \frac{\mu\,\beta-\mu-\beta}{\mu\,\beta}\right) \qquad (3)$$

Stability of Fixed Points: Using the processes of stability analysis, the following results emerged

- Eigenvalues corresponding to $P_1^*(0,\ 0)$ be $\lambda_1 = \mu$ and $\lambda_2 = 0$. Hence, P_1^* is stable if $0 < \mu < 1$.
- Eigenvalues corresponding to $P_2^*\left(\frac{\mu-1}{\mu},\ 0\right)$ are

  $\lambda_1 = \frac{\beta(\mu-1)}{\mu}$ and $\lambda_2 = 2 - \mu$

  Hence, P_2^* is stable if $\left|\lambda_1 = \frac{\beta(\mu-1)}{\mu}\right| < 1$ and $|\lambda_2 = 2 - \mu|<1$

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❖ Eigenvalues corresponding to $P_3^*\left(\frac{1}{\beta}, \frac{\mu\beta-\mu-\beta}{\mu\beta}\right)$ are

$$\lambda_1 = \frac{2\mu\beta-\mu^2+\sqrt{4\mu^2\beta^2+4\mu^3\beta-4\mu^3\beta^2+\mu^4}}{2\mu\beta},\ \lambda_2 = \frac{2\mu\beta-\mu^2-\sqrt{4\mu^2\beta^2+4\mu^3\beta-4\mu^3\beta^2+\mu^4}}{2\mu\beta}$$

Hence, P_3^* is stable if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.
For simplicity, we assume $\mu = \beta$ and proceed for further calculations.

3. Bifurcation Scenario:

We have obtained bifurcation diagrams for our prey-predator system by varying μ, $2.4 \le \mu \le 4.0$ and taking the initial point $(x_0, y_0) = (0,0)$. The bifurcation diagram is shown in the next slide.

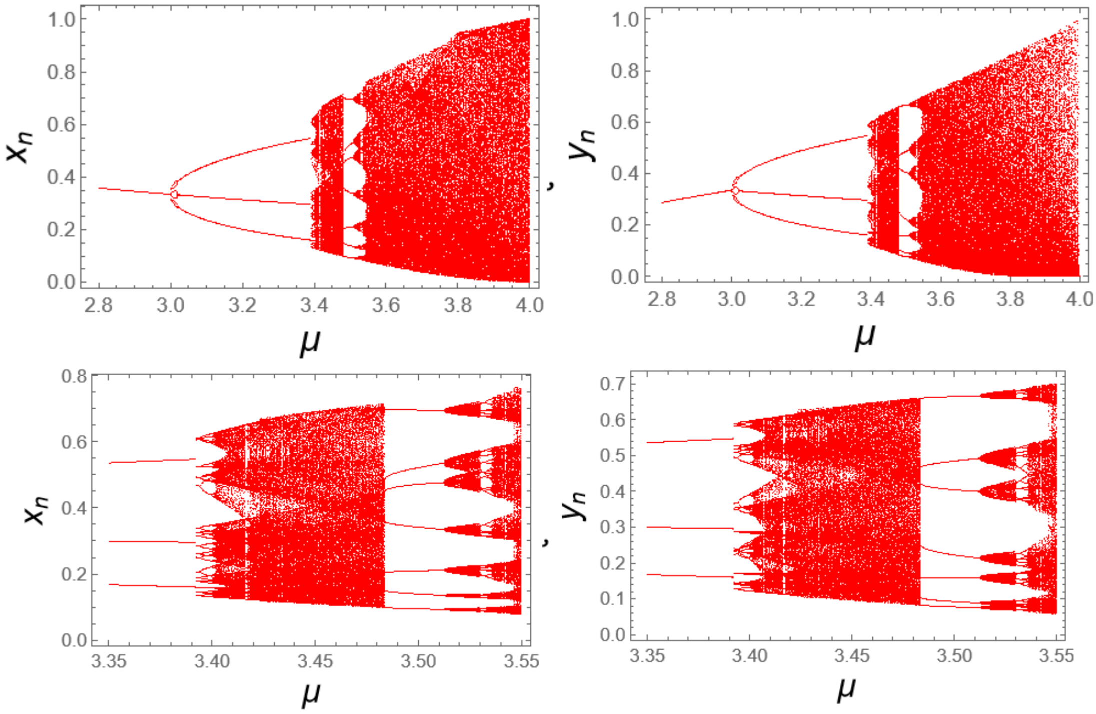

Figure 1: Bifurcation Scenario of the Prey-Predator System (1) for $2.4 \le \mu \le 4.0$. Lower figures correspond to magnifications near periodic windows when$3.35 \le \mu \le 3.55$.

Looking at the bifurcation diagram, as μ increases, we notice a very complex pattern of bifurcation.

❖ One periodic orbit splits into three periodic orbits, followed by chaos.

❖ Within the periodic window, we observe seven periodic orbits, each one again evolving into a period-doubling bifurcation, producing periodic windows with a complicated pattern.

❖ Looking into the lower figures, we see many more phenomena like bi-stability, intermittency, etc.

This indicates the system is complex, and such complexity is measured by topological entropy.

4. Results of Numerical Simulations

(α). Regular and Chaotic Attractors

As shown in Figure 2, regular and strange chaotic attractors are drawn for different values of parameterμ. We find a transition from regular to chaotic attractors with an increasing value of μ. In Figure 3, a plot of chaotic time series is shown, corresponding to the chaotic attractor.

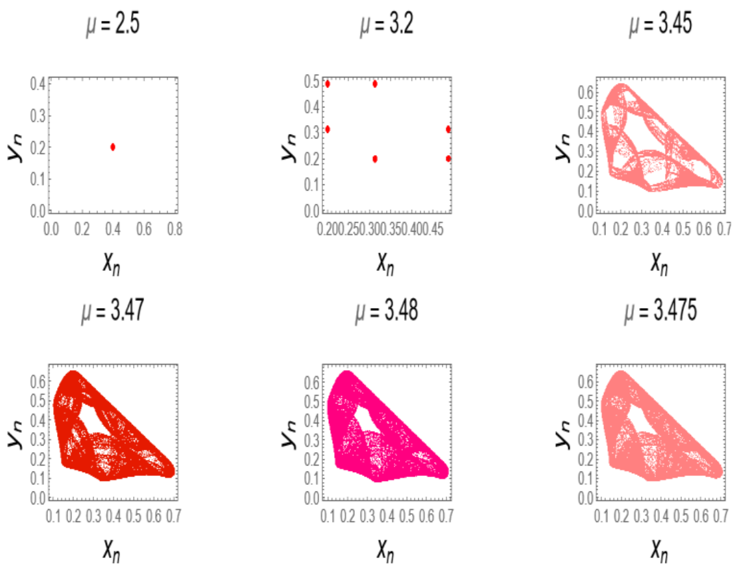

Figure 2: Plots of regular and chaotic (strange) attractors of system (1) for different increasing values of μ.

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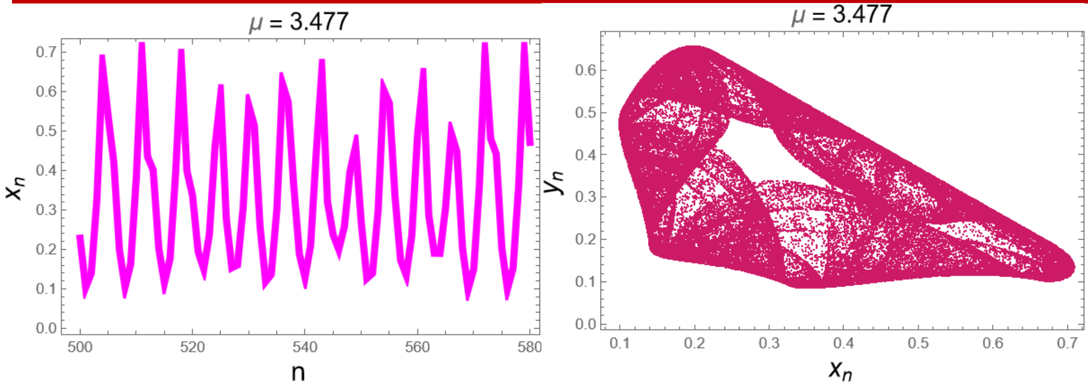

Figure 3: Chaotic time series plot and plot of corresponding strange chaotic attractor.

(b). Lyapunov Exponents (LCEs)

The state of a system is chaotic when it shows *sensitivity to initial conditions*, i.e., that the two orbits initiated extremely close to each other show divergence of behavior during long-term evolution. Lyapunov exponents provide a measure of the divergence of trajectories initiated closely, and if the system evolves chaotically, then LCEs > 0, and if the system evolution is regular, then LCEs < 0. In Figure 4, plots of Lyapunov exponents are shown

against cases of regular and chaotic attractors.

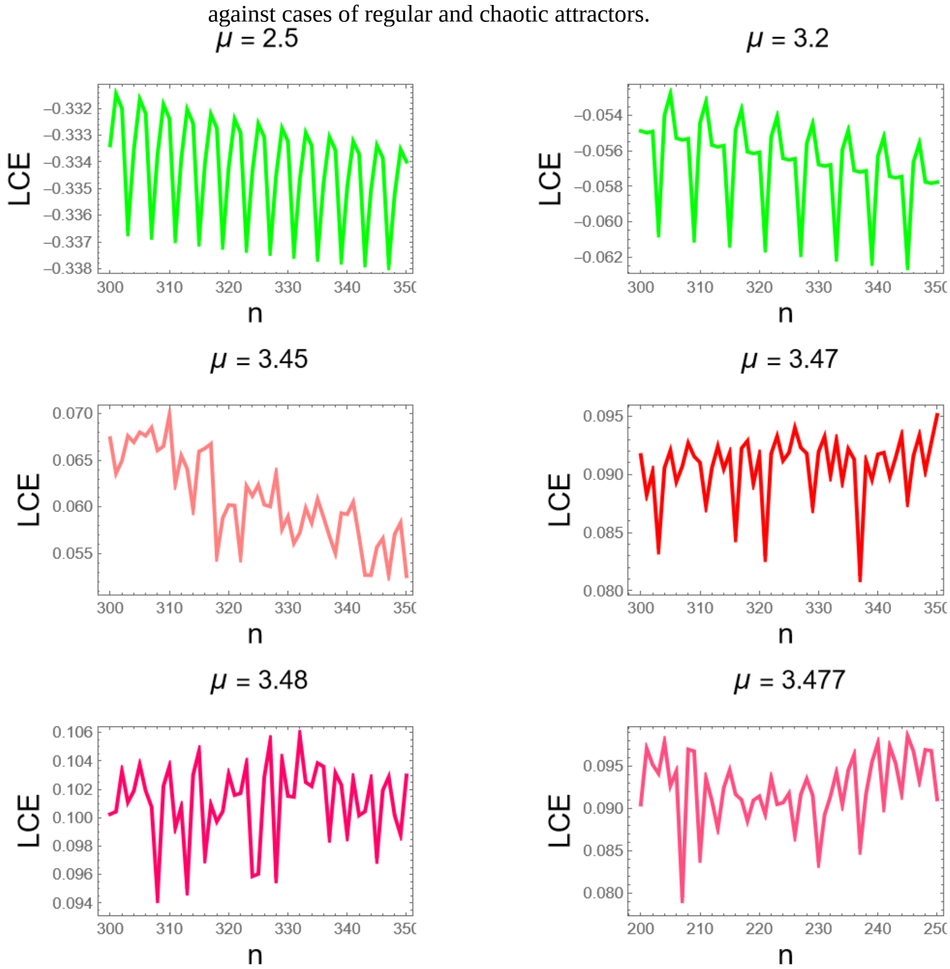

Figure 4: Lyapunov exponents (LCEs) plots for cases of regular and chaotic attractors.

(c) Topological Entropies

As mentioned in Section 1, the complexity of a complex system is measured by the increase (fluctuations) of topological entropy, and a greater increase in topological entropy signifies that it is a more complicated system. The measure of complexity involves the probabilistic law; it can be used to characterize the rate at which evolutions mix. Procedure of calculation of topological entropy follows:

Consider a finite partition of a state space X denoted by $P = \{A_1, A_2, A_3, \ldots, A_f\}$. Then a measure μ on X the total measure $\mu(X) = 1$ defines the probability of a given reading as

$$p_i = \mu(A_i), i = 1, 2, \ldots, f. \quad (4)$$

The partition's entropy is determined by

$$H(P) = -\sum_{i=0}^{f} p_i \, \mathrm{Log}\, p_i \quad (5)$$

As shown in Figure 5, plots of topological entropy for the prey-predator are shown for ranges $3.0 \le \mu \le 3.8$ and $3.52 \le \mu \le 3.56$. Actually, the lower figure is the magnification of the upper one, near the maximum increase in topological entropy. From these, one observes significant increases and fluctuations in topological entropy during evolution. This suggests that the system is complicated.

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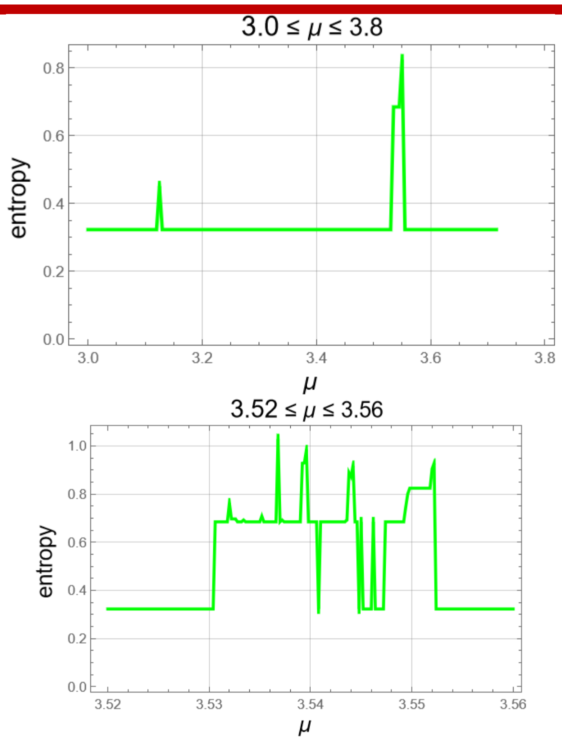
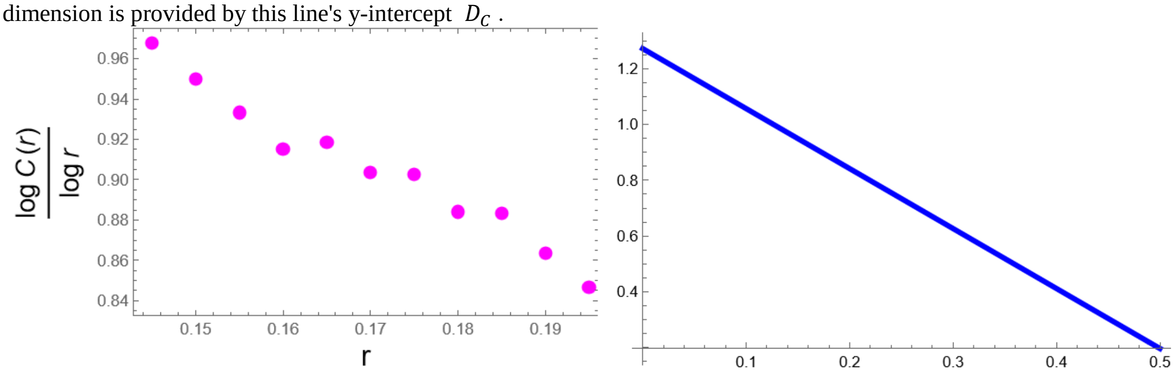


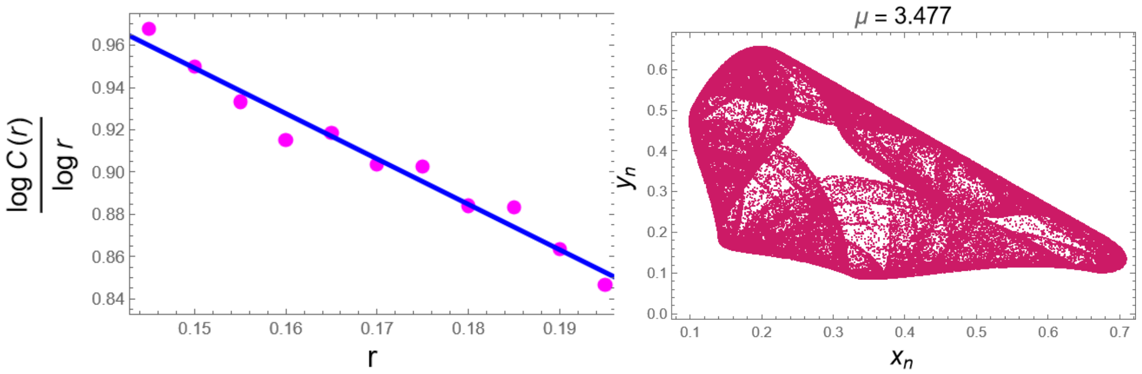
Figure 5: Plots of topological entropies for two different ranges of μ.

(d). Correlation Dimensions of Strange Chaotic Attractors

Fractal dimensions include the correlation dimension and it provides the measure of dimensionality of the chaotic attractor. All chaotic attractors (also sometimes called strange attractors) possess properties such as self-similarity and non-integer dimensions, as well as the correlation dimension. It is calculated statistically with the application of the Heaviside function. For this, one first has to calculate the correlation integral data C(r) for a certain r $\ll$ 1. Then, plot the correlation data curve by plotting $\frac{\log C(r)}{\log r}$ against r, as demonstrated in Figure 6. Then, to get the equation of the straight line fitting the data points by applying a linear fit criterion (linear regression) to the correlation data. The correlation dimension is provided by this line's y-intercept D_C .



y =1.272 – 2.149 x



$D_c \cong 1.27$

Figure 6: Plots to obtain correlation dimension of strange chaotic attractor

In the case of Figure 6, we considered the chaotic attractor obtained for $\mu = 3.477$. The correlation data curve's equation for the straight line fitting is therefore roughly found as

$$y = 1{,}272 - 2.149\, x\, . \tag{6}$$

The y-intercept of this curve is $\cong 1.722$.

Therefore, the correlation dimension of the above chaotic attractor is $D_C \cong 1.72$.
After repeating the procedure, the correlation dimensions of three more chaotic attractors were obtained and shown in Figure 7.

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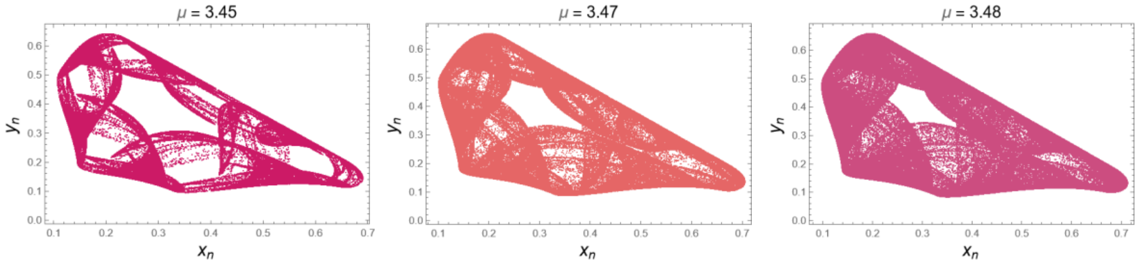

Figure 7: Plots of correlation dimensions of three more strange attractors.

5 . Concluding Remarks: We made the following observations:
The Prey-Predator System: Display Complexity property indicated by facts that:

❖ It displays a complex repeated pattern of bifurcation, and within chaotic regions of it, periodic windows appear of multiple periods, which again bifurcate into periods, doubling, and chaos-adding criteria.

Different periodic windows within bifurcation display mixed phenomena like bi-stability, intermittency, cascading consequences, the exhibit of hysteresis properties, etc.

❖ Plots of topological entropies, Figure 5, show significant increases and fluctuations of topological entropies for $3.0 \le \mu \le 3.5$, as clearly shown in $3.52 \le \mu \le 3.56$.

The study reveals that the composition of the prey-predator system is structurally multi-component, and these individual components develop in a self-sufficient manner, exhibiting complexity, chaos, and mixed nonlinearity features.

❖ The system also displays chaotic motion at some parameter spaces, and strange chaotic attractors are drawn (Figures 2–4).

❖ Chaotic attractors possess fractal properties like self-similarity and non-integer dimensions (in this case, correlation dimensions).

Most of the numerical simulation work is done by Mathematica 9.0, taking help from Martelli, M. [26] and Lynch, S. [25] to write proper codes. Graphical representations of results are quite significant and interesting.

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